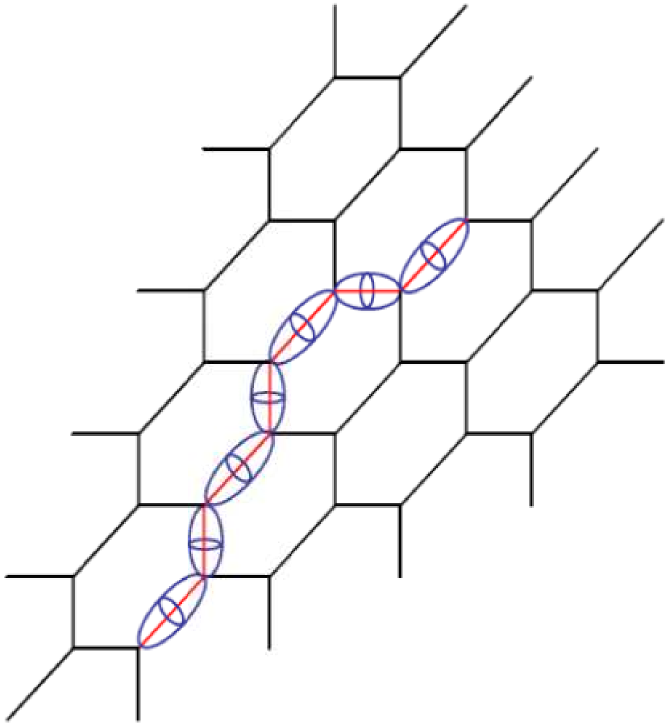
SYZ for affine A-type local Calabi-Yau manifolds

with Atsushi Kanazawa



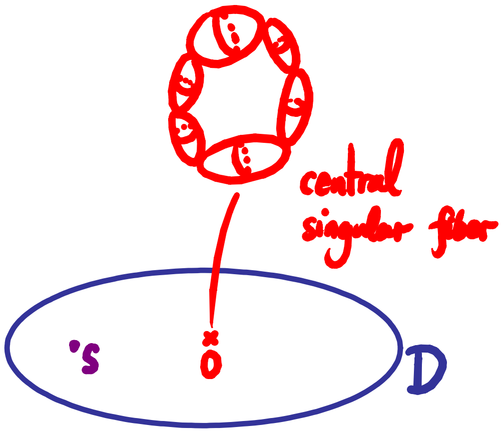
Siu-Cheong Lau

Boston University

Object of study

SYZ mirror of $\tilde{A}_{d-1}$ surface and their fiber products.

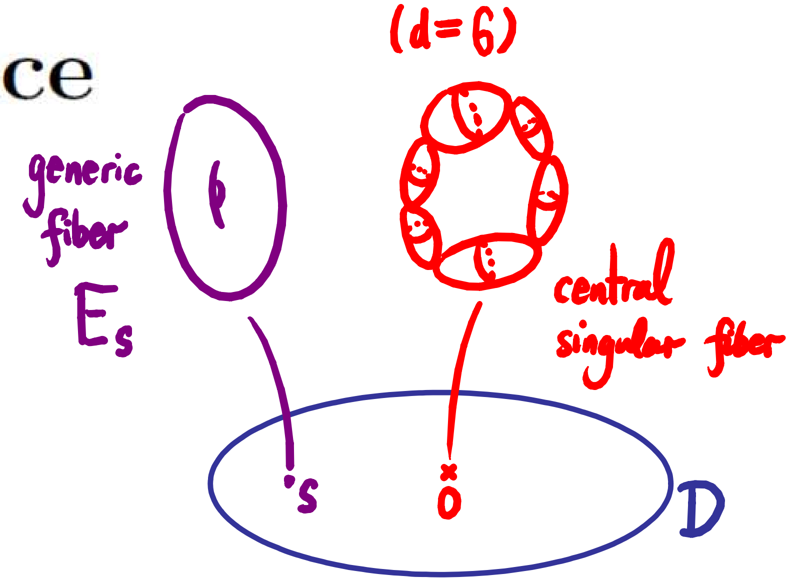
- The SYZ construction involves wall-crossing. ([Gross-Siebert], [Auroux])

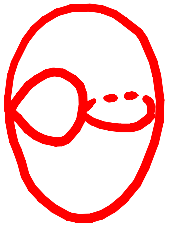


- Directly related with Yau-Zaslow formula for compact K3.
- They have interesting modular properties.
- Provides mirrors of general-type manifolds.

$\tilde{A}_{d-1}$ surface

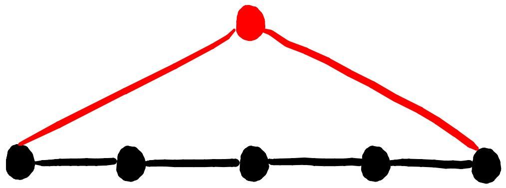
Elliptic fibration:



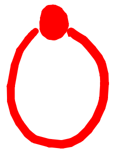


Affine Dynkin diagram:

$\tilde{A}_5$

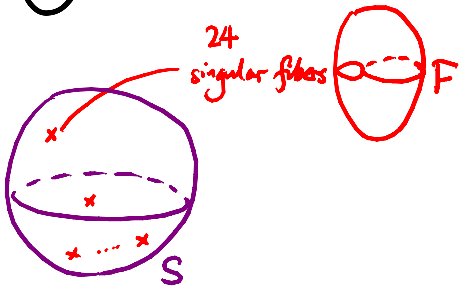


$\tilde{A}_0$

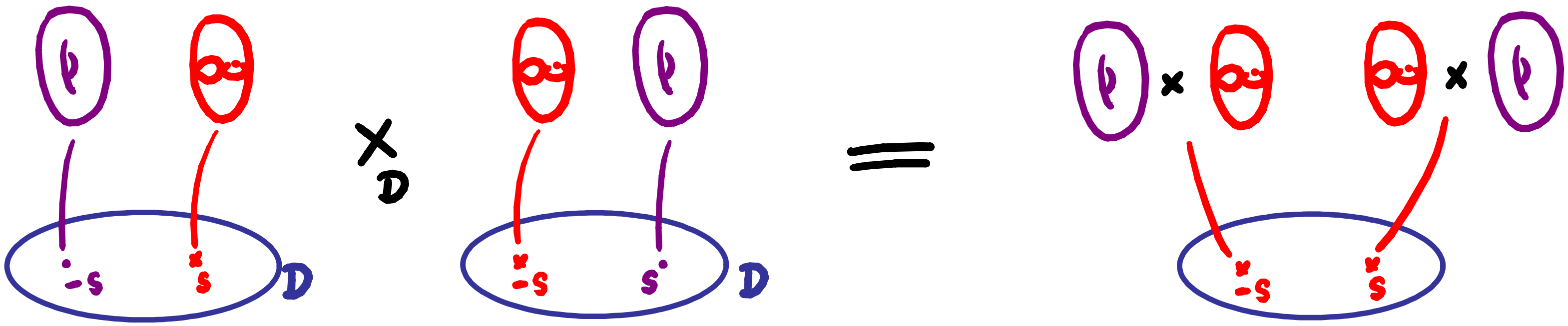


It is of type I_d in Kodaira classification of singular fibers.

A generic compact elliptic K3 has 24 singular $\tilde{A}_0$ fibers.



Fiber products of $\tilde{A}_0$ surfaces $\tilde{A}_0 \times_D \tilde{A}_0$



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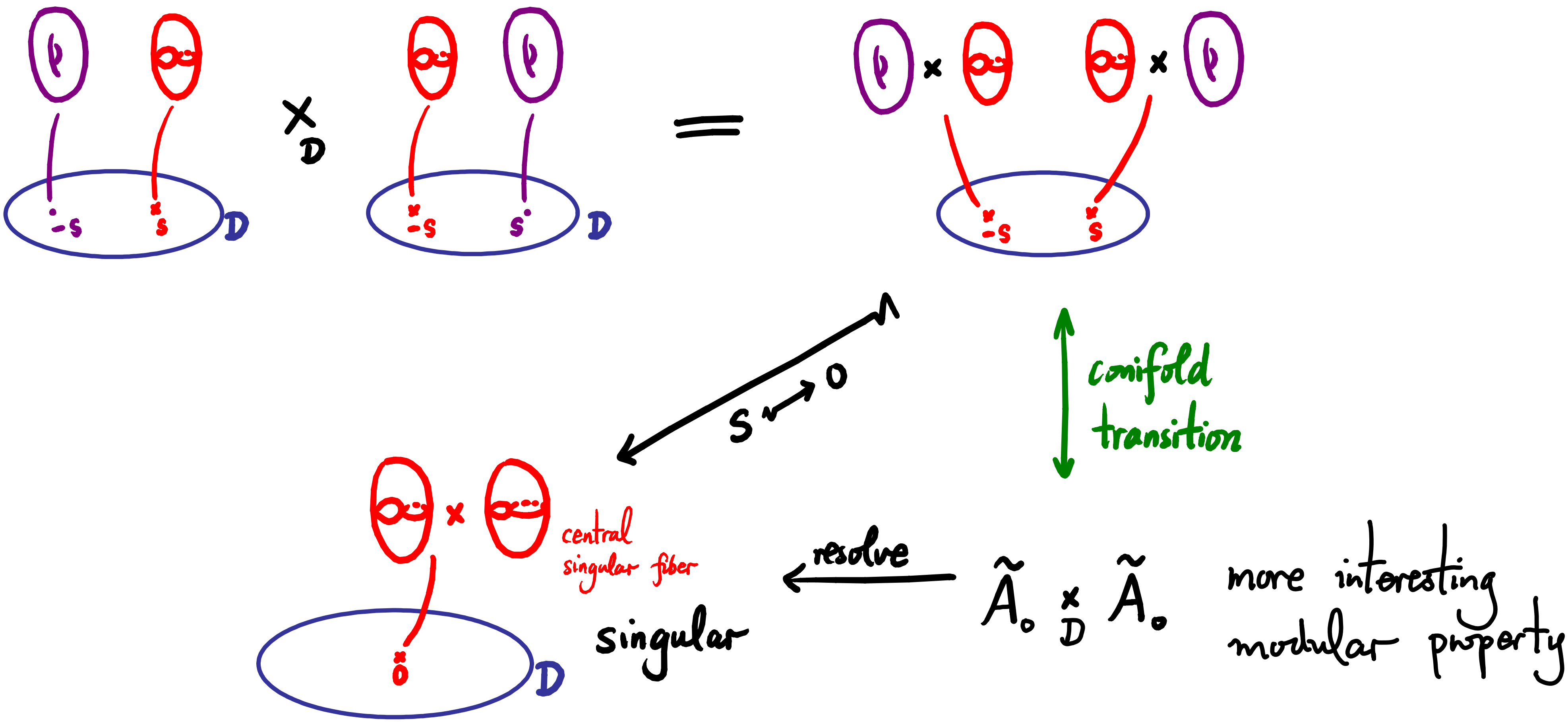

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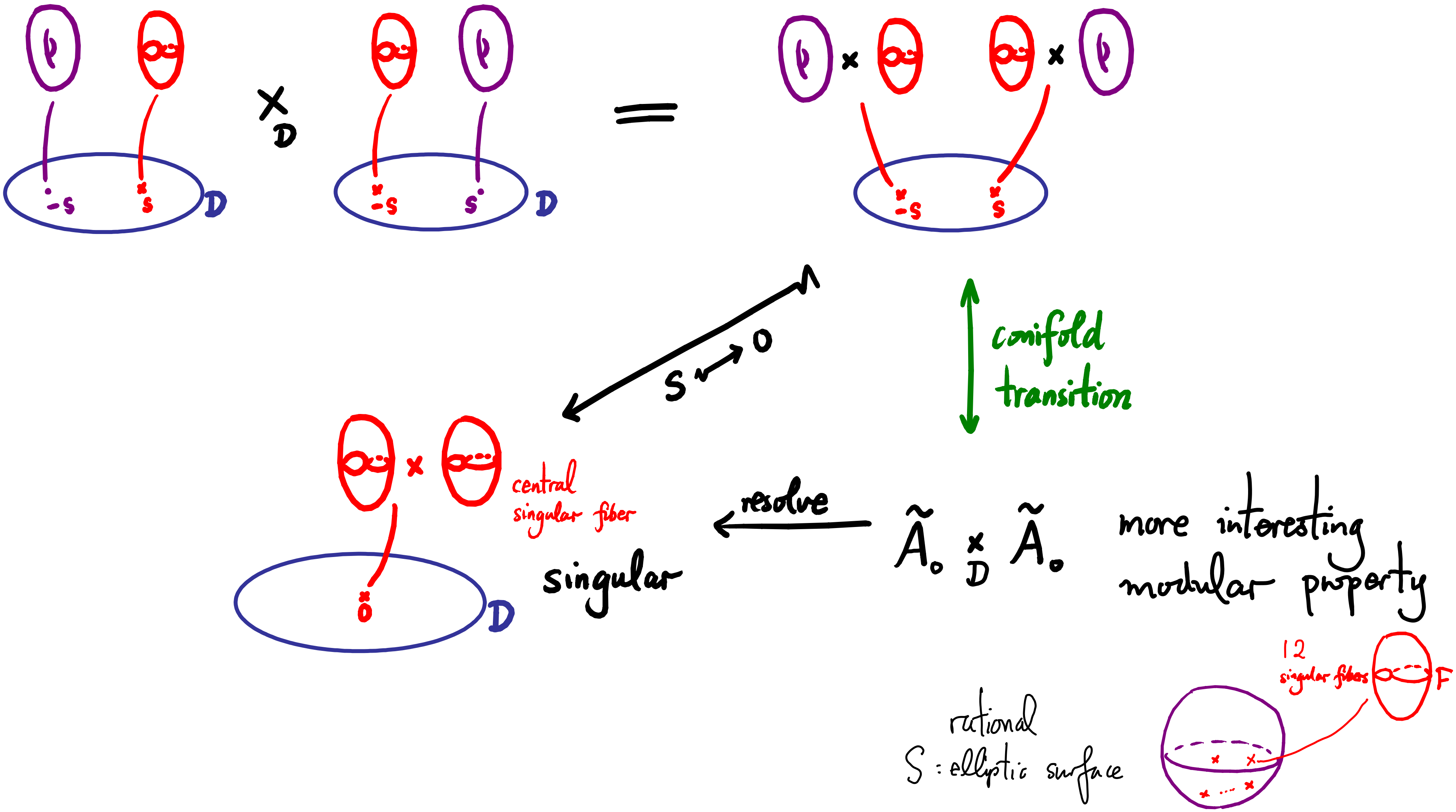


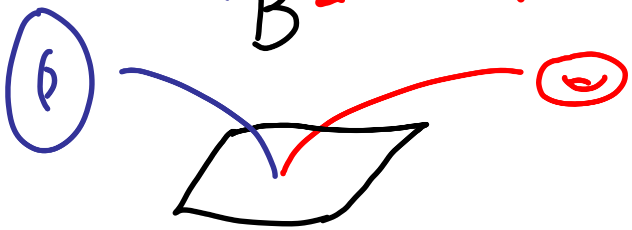
Local block of Shoen's Calabi-Yau threefold $\widetilde{S \times_{\mathbb{P}^1} S}$.

SYZ with quantum corrections

[Strominger - Yau - Zaslow]

$$\begin{tikzcd}
(X, \omega_q) \arrow[rr, leftrightarrow, "\text{dual}", "\text{tori}"'] \arrow[dr, "\text{Lagrangian torus fibration}"'] & & Y_0 \arrow[dl, "\text{dual torus fibration}"] \\
& B &
\end{tikzcd}$$

$Y_0 = \{(T, \nabla) : T$ is a smooth fiber of X, ∇ is a flat $U(1)$ connection$\}$



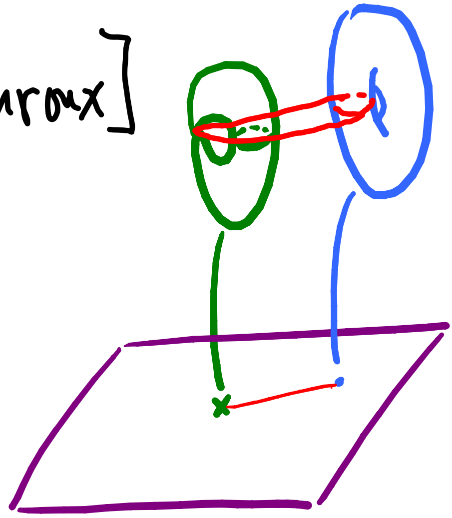
Y_0 has canonical complex coordinates.

e.g. $T^*\mathbb{R}^n/\mathbb{Z}^n \xleftrightarrow{\text{mirror}} T\mathbb{R}^n/\mathbb{Z}^n$

$\exists$ singular fibers! [Kontsevich-Soibelman, Gross-Siebert, Auroux]

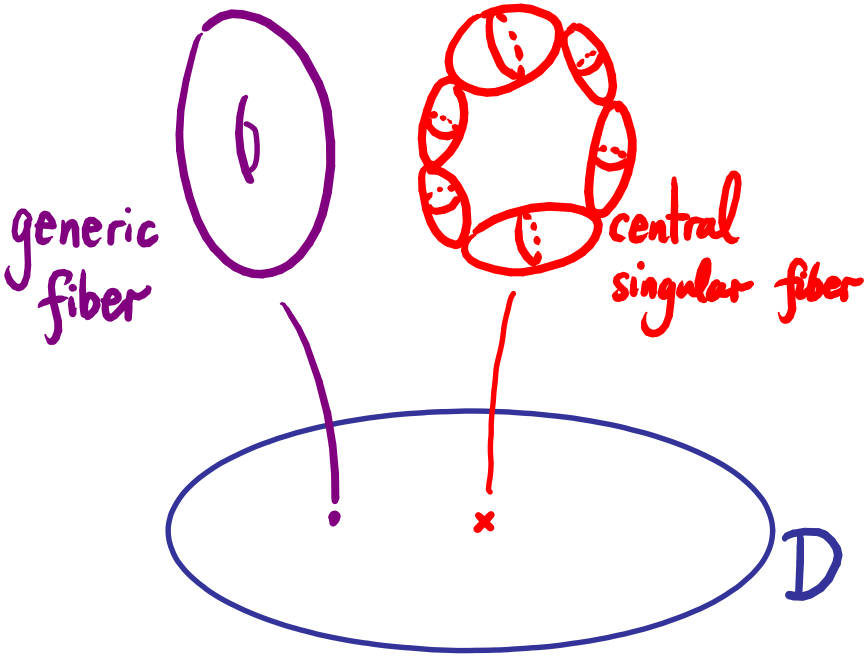
Complex coordinates are corrected by generating functions of open Gromov-Witten invariants

$\longrightarrow$ SYZ mirror Y_q



$\tilde{A}_{d-1}$ surface

Need: Kaehler structure and Lagrangian fibration



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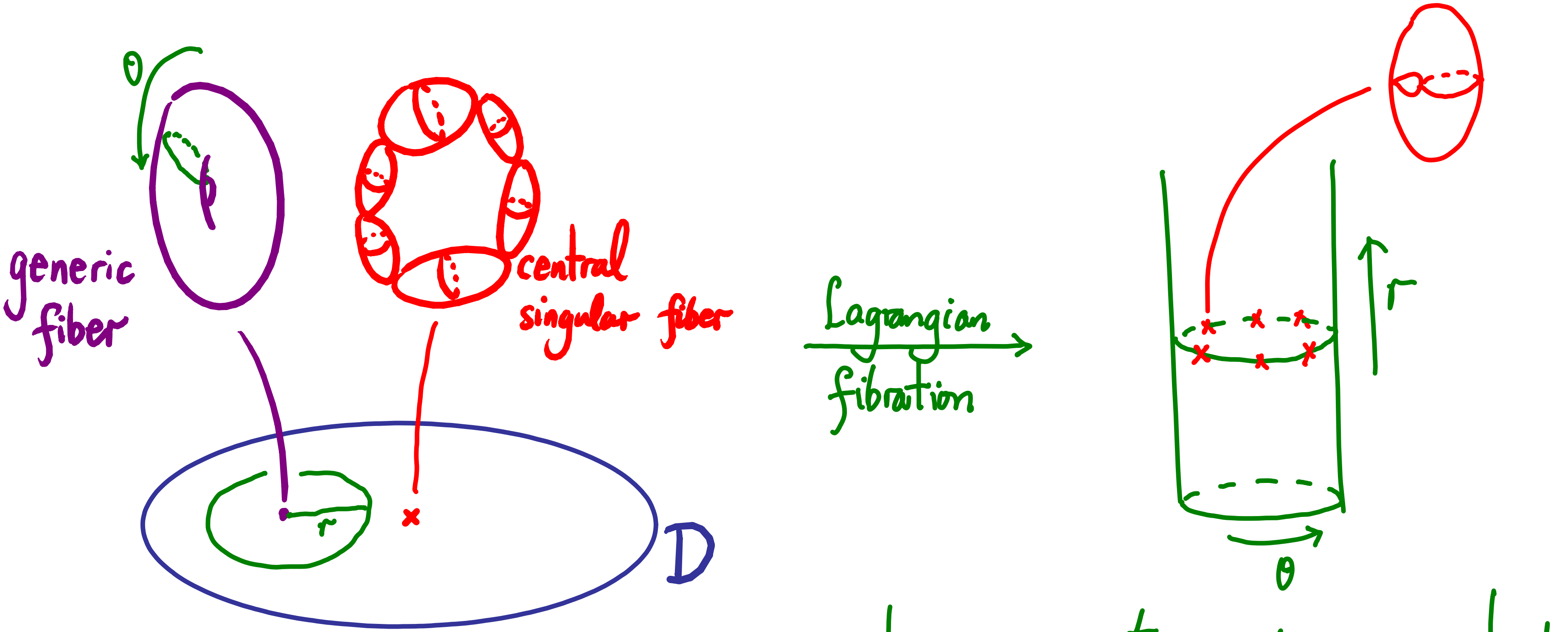
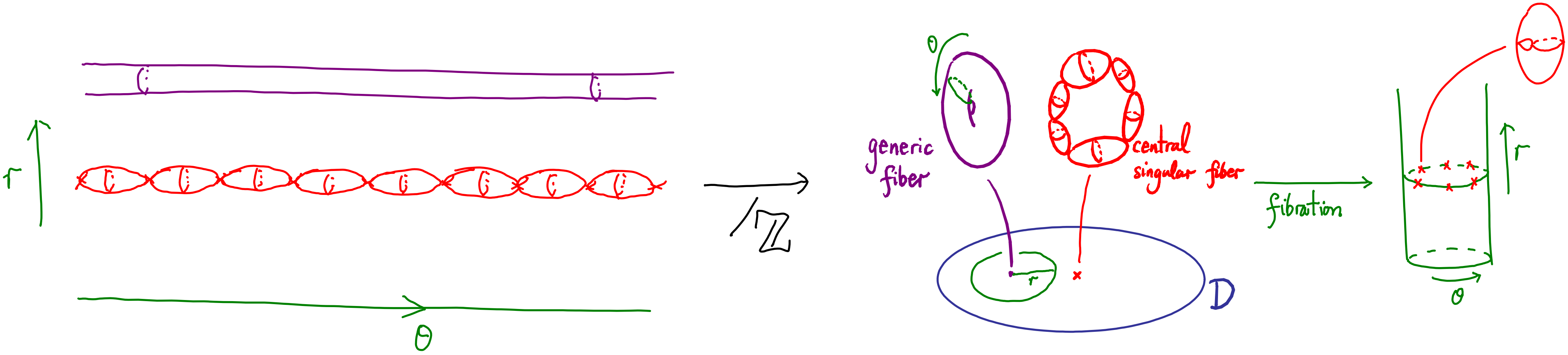


base is not simply connected.

Pull back the fibration to the universal cover of the base, construct the SYZ mirror upstairs, and then quotient out by Deck transformation group.

A toric realization upstairs [Mumford, Gross-Siebert]



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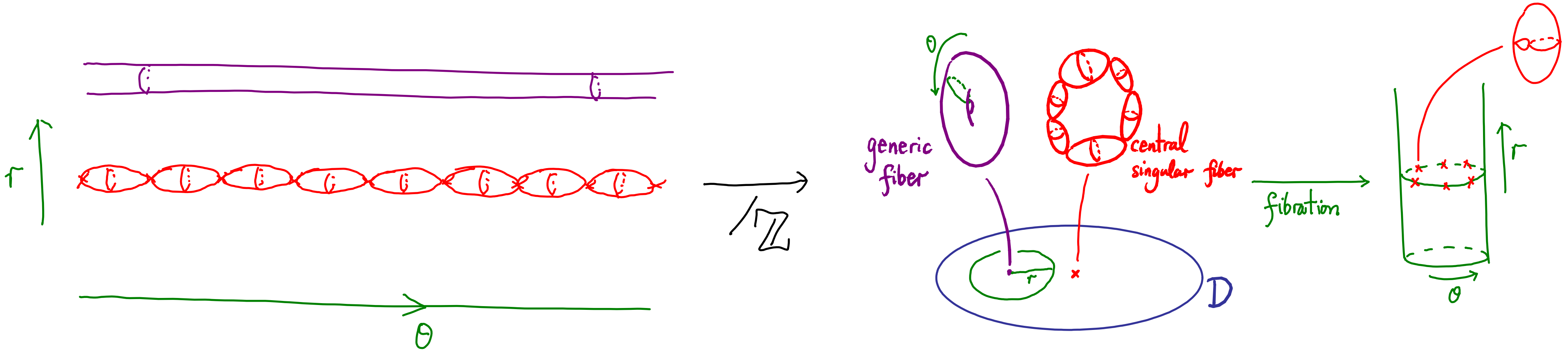
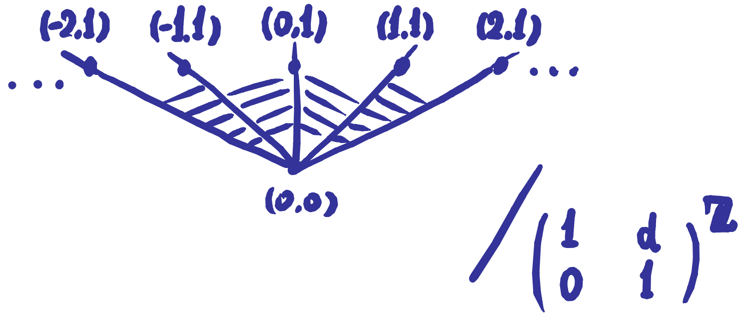
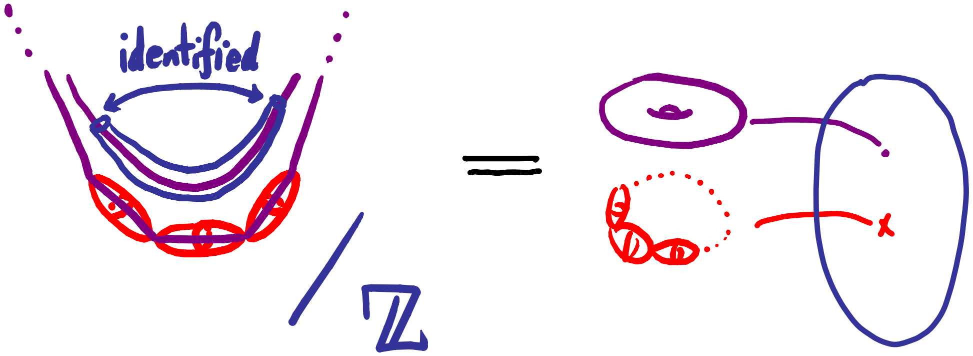


Realized by toric geometry: [Mumford, Gross-Siebert]

Fan



Polytope



These toric manifolds have infinite type!
They have infinitely many Kaehler parameters.

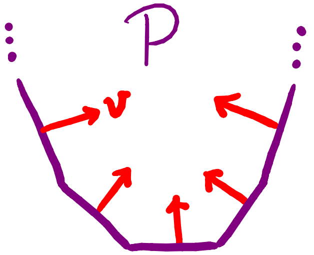
Kaehler metric for infinite-type toric manifolds

For toric manifolds,

Kähler potential can be taken to be

$$\frac{1}{2}\sum_{v\in\Sigma^{(1)}} \ell_v \cdot \log \ell_v \quad \text{on } P.$$

↑ ∞ sum! divergent!



$\ell_v = (v,\cdot) - c_v$ such that

$P = \bigcap_v \{\ell_v \geq 0\}.$

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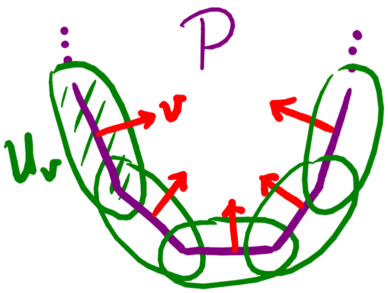


$\ell_v = (v,\cdot) - c_v$ such that

$$P = \bigcap_v \{\ell_v \geq 0\}.$$

Cut-off: Pick open sets U_v around facets of P s.t. $\forall p.\ \exists$ finitely many $U_v \ni p$.

Assume $\exists$ convex exhaustion of the fan to do this.

$$g \triangleq \frac{1}{2}\sum_{v\in\Sigma^{(1)}} \rho_v\cdot\ell_v\cdot\log \ell_v$$ defined on a neighborhood of ∂P in P

↳ supported in U_v.

$\Rightarrow$ toric Kähler metric $\partial\bar{\partial} g$ on a neighborhood of toric divisors.

Kaehler metric on group quotient

A toric neighborhood of toric divisors such that G acts freely

Suppose $G \underset{\text{free}}{\curvearrowright} (N, \Sigma - \{0\}) \longrightarrow X_{\Sigma}^{0}/G$.

Kähler metric on X_{Σ}^{0}/G ?

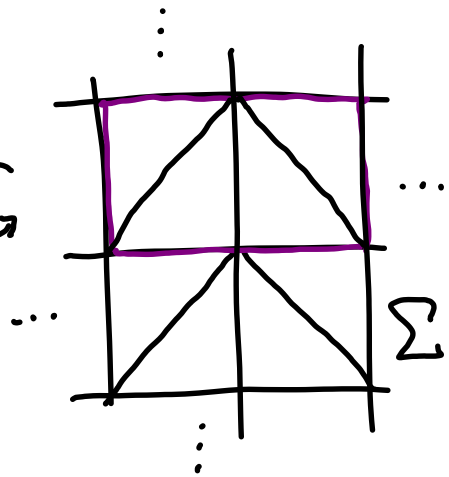
Kaehler metric on group quotient

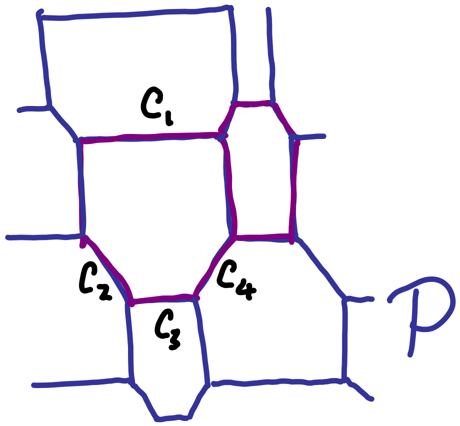
Suppose $G \underset{\text{free}}{\curvearrowright} (N, \Sigma - \{0\}) \longrightarrow X^{0}_{\Sigma} / G$.

A toric neighborhood of toric divisors such that G acts freely

Note: X^{0}_{Σ} may not have toric invariant Kähler metric!

e.g. $\mathbb{Z}^2 \curvearrowright$





$C_1 \sim C_2 + C_3 + C_4$

but $(0,1) \cdot C_3 = C_1$!

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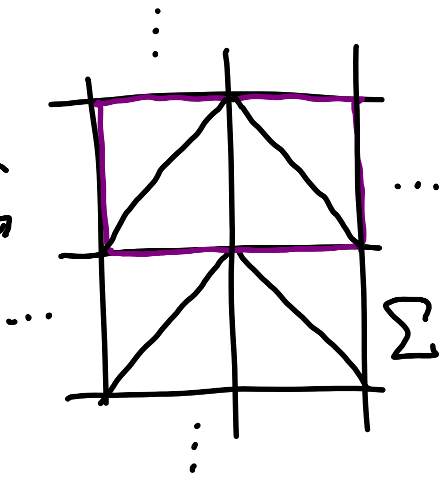

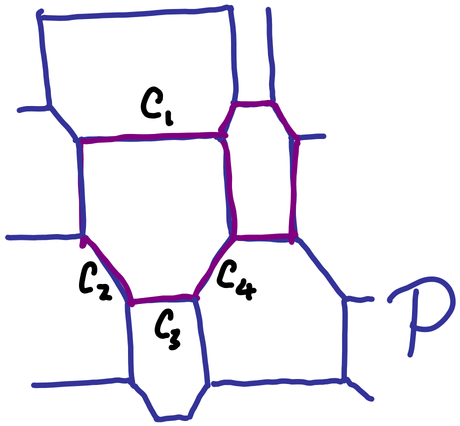

$C_1 \sim C_2 + C_3 + C_4$

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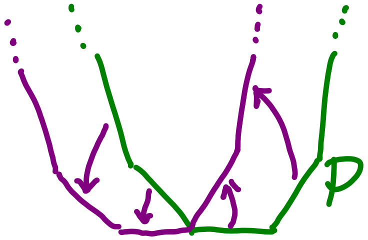
Prop.: Assume that P is invariant under $G \curvearrowright M_{\mathbb{R}}$ up to translation. Then $\exists$ G-inv. toric metric.

e.g.



$\circlearrowright \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Kaehler moduli for infinite-type toric manifolds

Every holomorphic curve is homologous to toric invariant curves.

Let $\{C_i : i \in \mathbb{Z}_{\geq 0}\}$ be the set of irreducible toric invariant curves. (∞ set.)

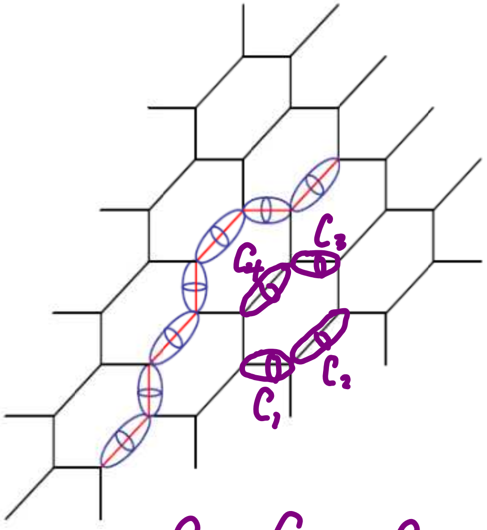
require only finitely many terms are identified under I

$$\mathcal{M}^{\text{Käh}}_{X^o} = \operatorname{Spec}\left(\mathbb{C}[\![q^{C_1}, q^{C_2}, \ldots\ldots]\!]^{f} \big/ I \right)$$

(∞ dim.)

where I is gen. by homology relations among C_i.

$$\mathcal{M}^{\text{Käh}}_{X^o/G} \ ?$$



$$C_1 + C_2 \sim C_3 + C_4$$

$$\Rightarrow q^{C_1} q^{C_2} = q^{C_3} q^{C_4}.$$

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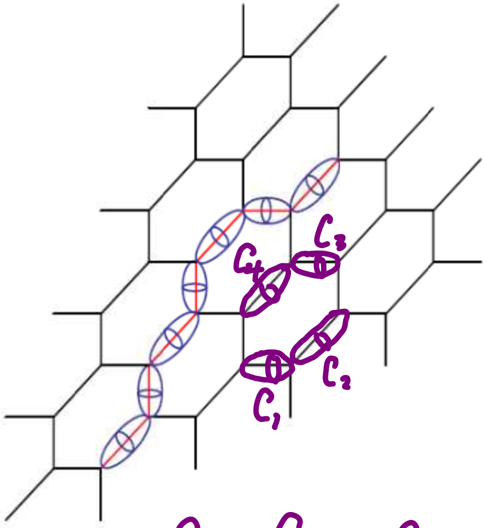

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$$C_1 + C_2 \sim C_3 + C_4$$

$$\Rightarrow q^{C_1} q^{C_2} = q^{C_3} q^{C_4}.$$

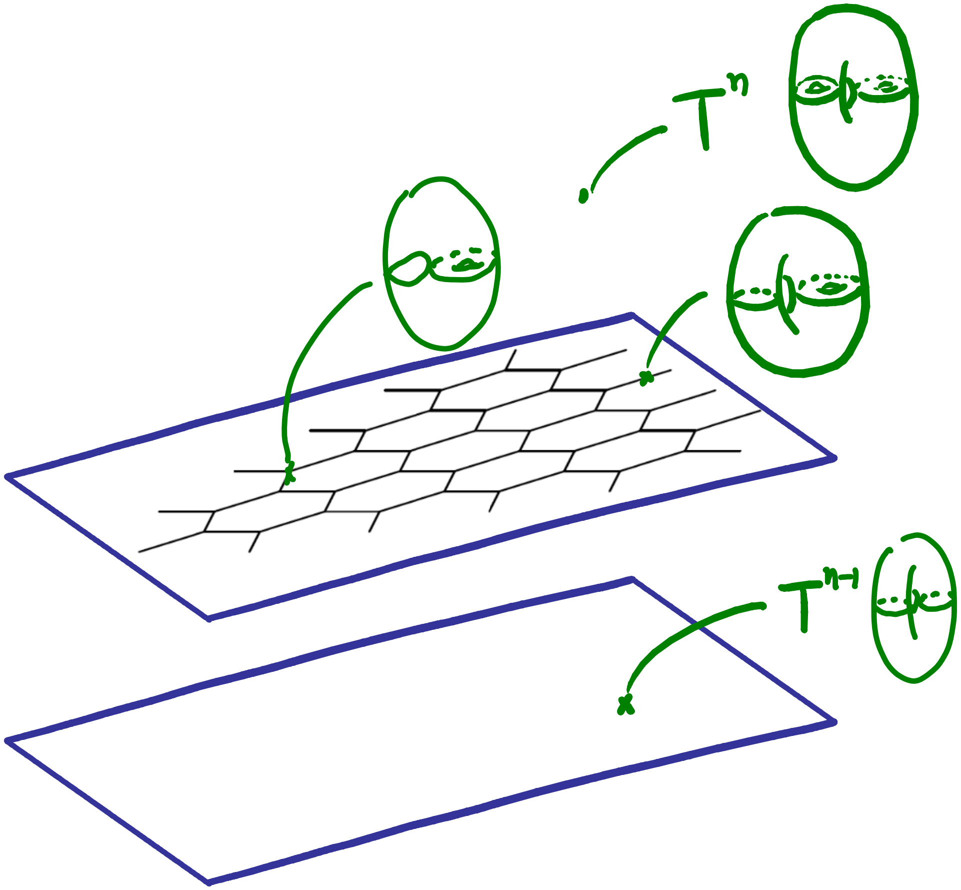
$$\mathcal{M}^{\text{Käh}}_{X^{\circ}/G} = \operatorname{Spec}\left(\mathbb{C}[\![\{q^{C_1}, q^{C_2}, \ldots\}/G]\!]^{f} \big/ (I/G)\right)$$

(can be finite dim.)

where $g \cdot q^{C_i} = q^{C_i \cdot g^{-1}}$.

Lagrangian fibration on infinite-type toric CY

$T^{\perp\nu} \subset T^n \circlearrowleft (X^0_\Sigma, \omega) \xrightarrow[\mu]{} \mathbb{R}^n \xrightarrow[/\mathbb{R}\cdot\nu]{} \mathbb{R}^{n-1}$ moment map.



([Harvey-Lawson], [Gross], [Goldstein])

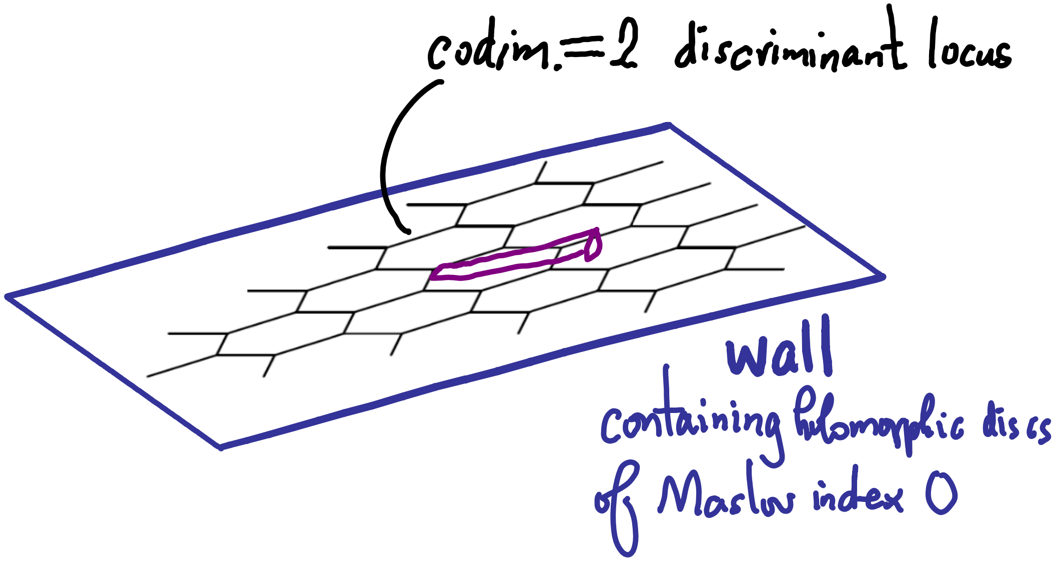
$$\begin{tikzcd} X^0 \arrow[d, "{(\bar{\mu}, |z^\nu - \varepsilon|)}"] \\ \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \end{tikzcd}$$

The Lagrangian fibration descends to quotient by G.
Use this to construct the SYZ mirror.

Quantum correction: wall-crossing of open GW

Semi-flat mirror: take the dual torus fibration away from the singular fibers [Leung-Yau-Zaslow].

Wall-crossing:



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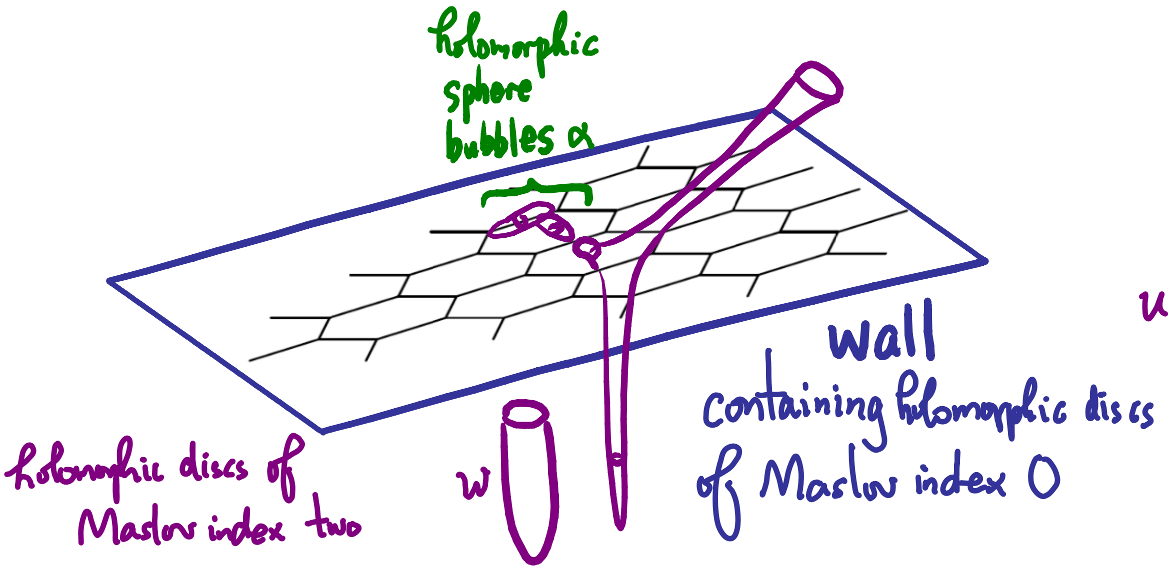

$$u = \begin{cases} W \cdot F^{open}(Z;q) & \text{above the wall} \\ W & \text{below the wall} \end{cases}$$

$$F^{open} = \sum_{v \in \Sigma^{(1)}} \Big(\sum_{\alpha \in H_2^{eff}} \underbrace{n_{\beta_v + \alpha}}_{\text{counting of stable discs in class } \beta_v + \alpha} q^{\alpha} \Big) q^{\beta_v - \beta_0} Z^{v}.$$

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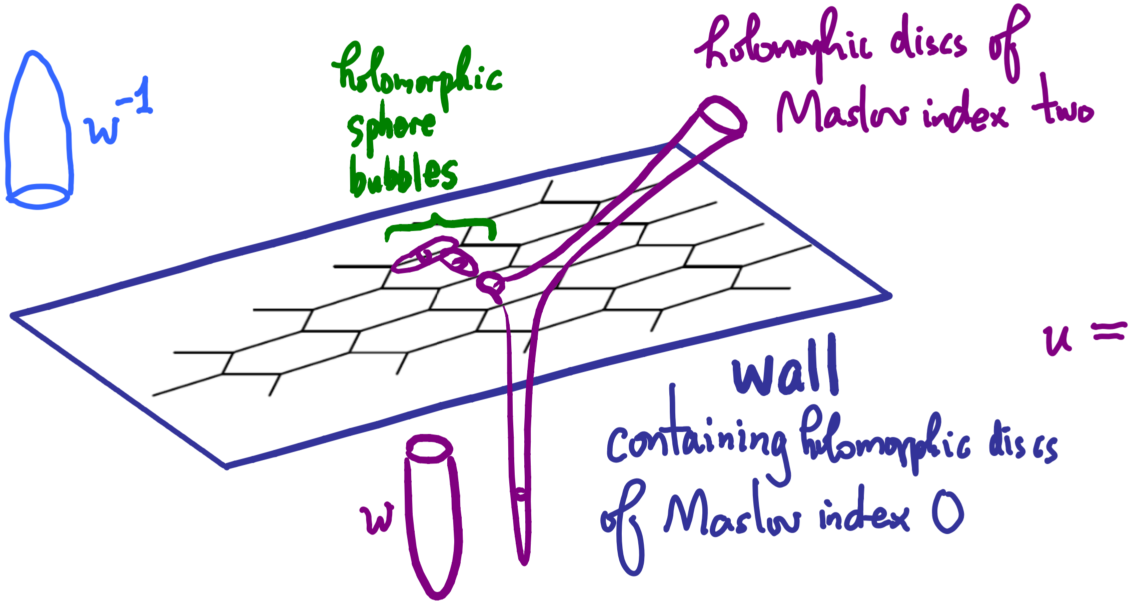
$$v = \begin{cases} w^{-1} \\ w^{-1}\cdot F^{\text{open}}(z,q) \end{cases}$$



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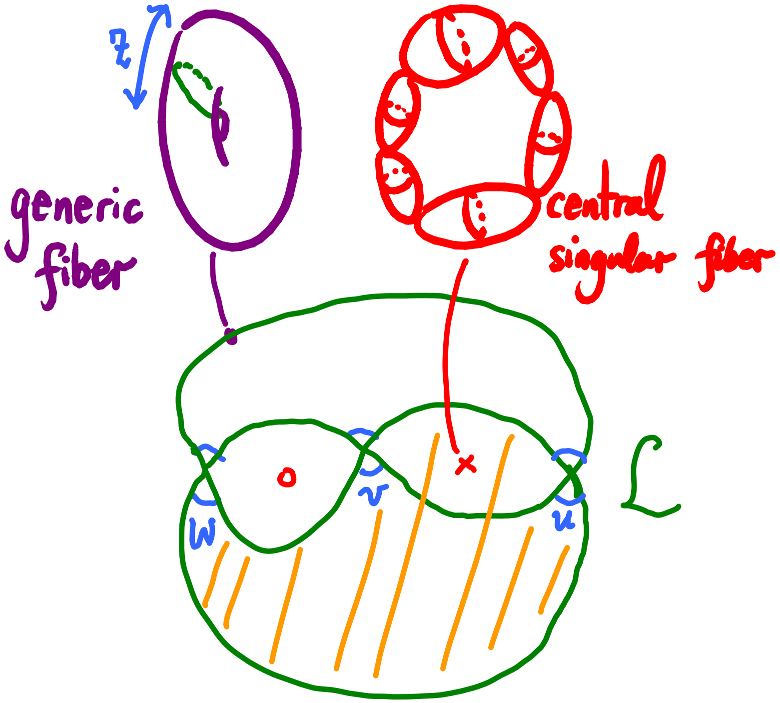
[Chan-L.-Leung]

The SYZ mirror is $Y_q \triangleq \{(u,v,z)\in\mathbb{C}^2\times(\mathbb{C}^\times)^{n-1} : uv = F^{\text{open}}(z;q)\}$.

Remark: wall-crossing can be captured by immersed Lagrangian Floer theory



Count stable polygons bounded by $\mathcal{L}$.

$$W = wuv + w \cdot F^{open}(q; z) \text{ on } \mathbb{C}^3 \times (\mathbb{C}^\times)^{n-1} \ni (w, u, v, z)$$

$$\mathrm{Crit}(W) = Y_q \subset \mathbb{C}^3 \times (\mathbb{C}^\times)^{n-1}.$$

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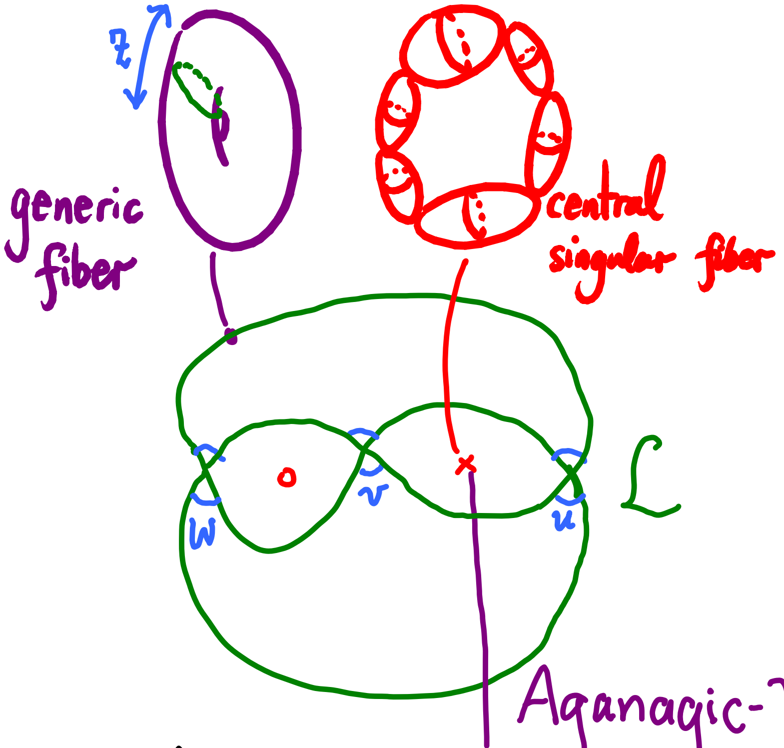

Aganagic-Vafa Lagrangian brane ⟷ matrix factorization of W

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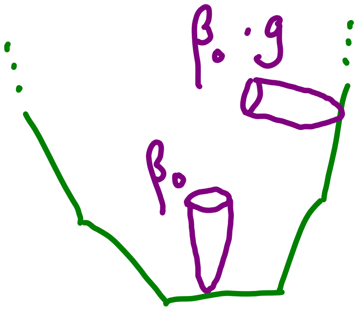
G action on SYZ mirror

$G \curvearrowright X^\circ_\Sigma$. **There is a natural induced G action on Y.**

Prop.: $F^{\text{open}}(g\cdot\vec{z};q) = q^{-(\beta_0\cdot g-\beta_0)}\, F^{\text{open}}(\vec{z};q).$



$$g\cdot u = \begin{cases} q^{-(\beta_0\cdot g-\beta_0)}\cdot u & \text{above the wall} \\ u & \text{below the wall} \end{cases}$$

$$g\cdot v = \begin{cases} v & \text{above the wall} \\ q^{-(\beta_0\cdot g-\beta_0)}\cdot v & \text{below the wall} \end{cases}$$

Hence G preserves $uv = F^{\text{open}}(z;q)$.

The SYZ mirror of X°/G is given by

$$\{(u,v,z)\in\mathbb{C}^2\times(\mathbb{C}^\times)^{n-1} : uv = F^{\text{open}}(z;q)\}/G$$

where $F^{\text{open}} = \sum_{v\in\Sigma^{(1)}}\Big(\sum_{\alpha\in H_2^{\text{eff}}} n_{\beta_v+\alpha}\, q^{\alpha}\Big)\, q^{\beta_v-\beta_0}\, z^{v}.$

GKZ system and mirror map for infinite-type toric

GKZ system: $\Box_d \cdot h = 0 \quad \forall d \in \underbrace{H_2}_{\infty \text{ dim.}}$ for $h \in \mathbb{C}[[y_1, \dots]]^f / I$,

where $\Box_d := \prod_{i:(D_i,d)>0} \prod_{k=0}^{(D_i,d)-1} (\hat{D}_i - kz) - y^d \prod_{i:(D_i,d)<0} \prod_{k=0}^{-(D_i,d)-1} (\hat{D}_i - kz).$

Coefficients of

$$\mathbf{I}(z;y) := e^{z^{-1}\sum_{l=1}^{\infty} T_l \log y^{\alpha_l}} \mathbf{I}_{\text{main}}(z;y) := e^{z^{-1}\sum_{l=1}^{\infty} T_l \log y^{\alpha_l}} \sum_{d \in H_2^{\text{eff}}(X,\mathbb{Z})} y^d \prod_i \frac{\prod_{m=-\infty}^{0}(D_i + mz)}{\prod_{m=-\infty}^{d \cdot D_i}(D_i + mz)}.$$

satisfy the GKZ system.

(∞ components of D_i)

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satisfy the GKZ system. (∞ components of D_i)

The mirror map is defined as $1/z$-coeff. of I

$\mathbb{C}[[y_1, \dots]]^f / I$

$\rotatebox{90}{$\in$}$

$= \mathrm{Id} - \sum_{v \in \Sigma^{(1)}} h_v(y) [D_v]$. $q^c(y) = y^c \exp\left(-\sum_v (C \cdot D_v) h_v(y)\right).$

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$\rotatebox{90}{$\in$}$

$= \text{Id} - \sum_{v \in \Sigma^{(1)}} h_v(y)\, [D_v]$. $q^c(y) = y^c \exp\left(-\sum_v (C \cdot D_v)\, h_v(y)\right)$.

Lemma: $g \cdot \mathbb{I} = \mathbb{I}$. Hence $g \cdot q^c(y) = q^{c \cdot g^{-1}}(y)$.

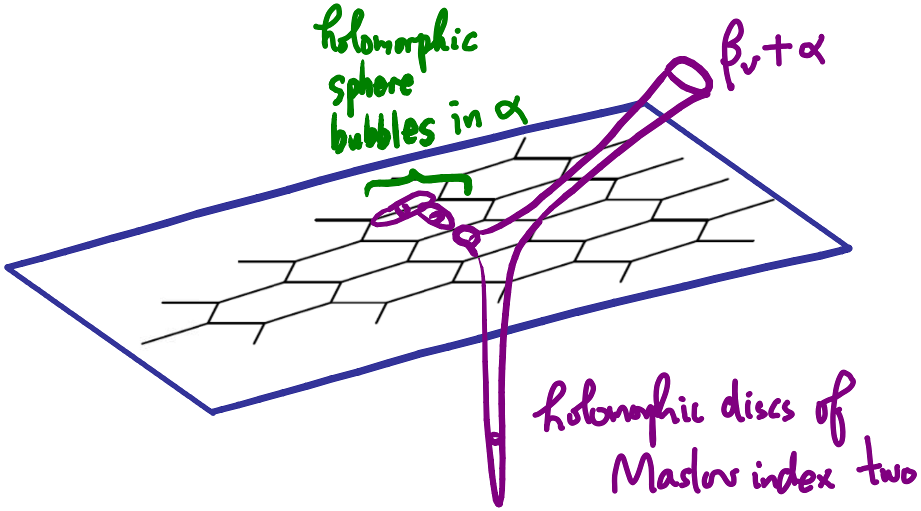
$$G \circlearrowright \left(\mathcal{M}_Y^{cpx} \xrightarrow[\text{map}]{\text{mirror}} \mathcal{M}_X^{Käh}\right) \Rightarrow \mathcal{M}_{Y/G}^{cpx} \xrightarrow[\text{map}]{\text{mirror}} \mathcal{M}_{X/G}^{Käh}.$$

Open mirror theorem for infinite-type toric CY

$$n_{\beta_v+\alpha} \triangleq \int_{[\mathcal{M}_1(\beta_v+\alpha)]^{\text{virt.}}} ev_1^* [pt]$$

[Fukaya-Oh-Ohta-Ono]

$$ev_1 : \mathcal{M}_1(\beta_v+\alpha) \longrightarrow T \subset X^\circ_\Sigma .$$



Theorem [Chan-Cho-L.-Leung] extended to ∞-type toric C.Y.:

$$\varphi_v \triangleq \sum_{\alpha\in H_2^{\text{eff}}} n_{\beta_v+\alpha}\, q^\alpha = \exp h_v(y(q)).$$

inverse mirror map

'hypergeometric' function in ∞ variables

Open mirror theorem for infinite-type toric CY

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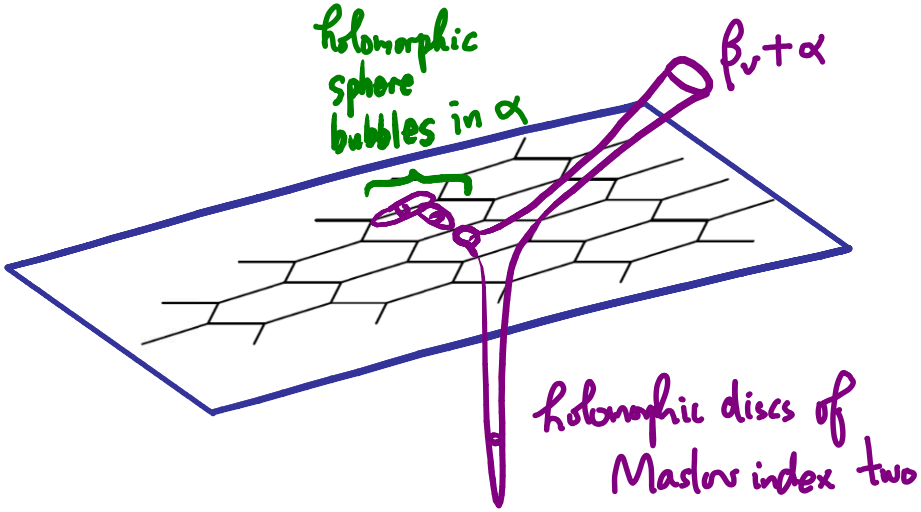


<u>Theorem</u> [Chan-Cho-L.-Leung] extended to ∞-type toric C.Y.:

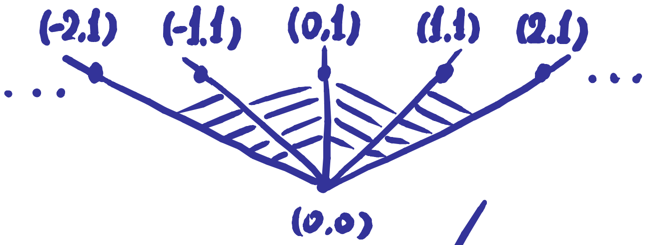
$$\varphi_v \triangleq \sum_{\alpha\in H_2^{\mathrm{eff}}} n_{\beta_v+\alpha}\, q^\alpha = \exp \underbrace{h_v}_{\text{'hypergeometric' function in } \infty \text{ variables}}(\overbrace{y}^{\text{inverse mirror map}}(q)).$$

G-invariance: $n_{(\beta_v+\alpha)\cdot g} = n_{\beta_v+\alpha}$ for $g \in G \curvearrowright X_\Sigma^\circ$.

$$\varphi_{v\cdot g} = g^{-1}\cdot \varphi_v .$$

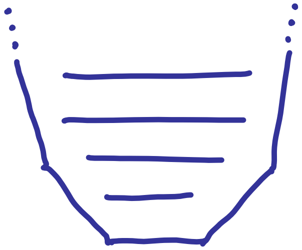
SYZ mirror of $\tilde{A}_0$ surface

fan



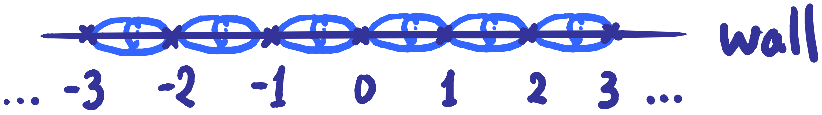
$$/\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{\mathbb{Z}}$$

polytope



$$/\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{\mathbb{Z}}$$

Lagrangian fibration



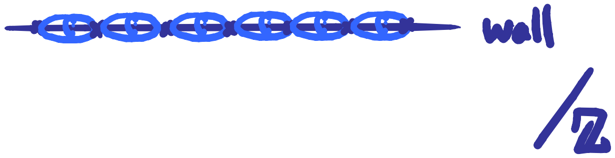
$$/\mathbb{Z}$$

SYZ mirror : $uv = F^{\text{open}}$ where (only one Kähler parameter q after quotient)

$$F^{\text{open}} = \sum_{l=-\infty}^{\infty}\Big(\sum_{\alpha\in H_2^{\text{eff}}} n_{\beta_l+\alpha}\, q^{\alpha}\Big)\, q^{\beta_l-\beta_0}\, z^{l} = \Big(\sum_{\alpha\in H_2^{\text{eff}}} n_{\beta_l+\alpha}\, q^{\alpha}\Big)\sum_{l=-\infty}^{\infty} q^{\beta_l-\beta_0}\, z^{l}.$$

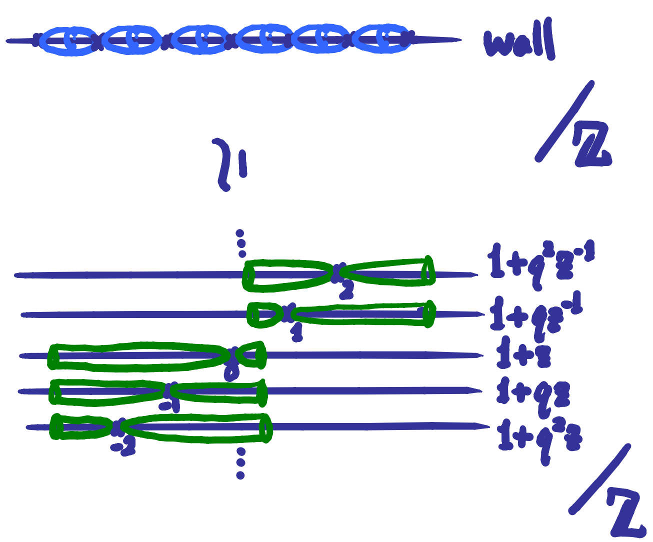
SYZ mirror of $\tilde{A}_0$ surface

$$F^{open} = \left(\sum_{\alpha \in H_2^{eff}} n_{\beta_\ell + \alpha} \, q^{\alpha}\right) \sum_{\ell=-\infty}^{\infty} q^{\beta_\ell - \beta_0} z^{\ell}$$

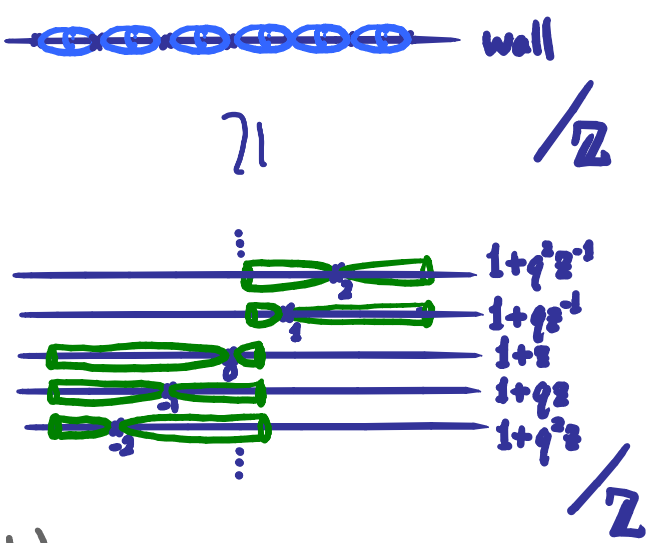


SYZ mirror of $\tilde{A}_0$ surface

$$F^{open} = \prod_{j=1}^{\infty}(1+q^j z^{-1})\cdot\prod_{k=0}^{\infty}(1+q^k z)$$



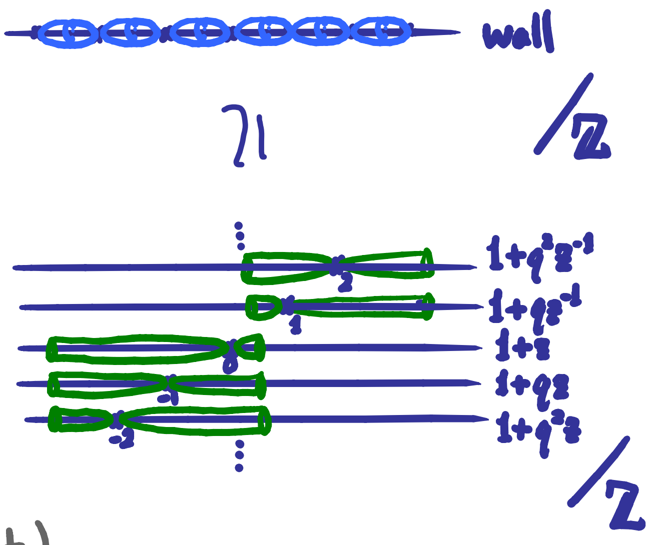
SYZ mirror of $\tilde{A}_0$ surface



$$F^{open} = \prod_{j=1}^{\infty}(1+q^j z^{-1}) \cdot \prod_{k=0}^{\infty}(1+q^k z)$$

$$= \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right) \cdot \sum_{\ell=-\infty}^{\infty} q^{\frac{\ell(\ell-1)}{2}} z^{\ell} \quad \text{(Jacobi triple product)}$$

SYZ mirror of $\tilde{A}_0$ surface



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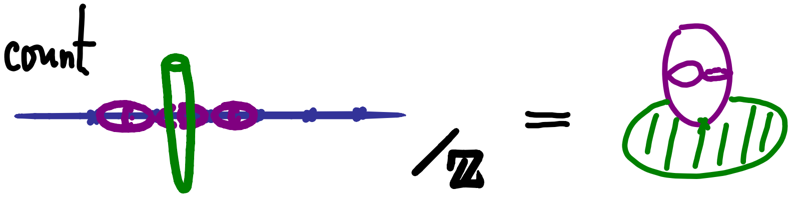
$$= \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right) \cdot \sum_{\ell=-\infty}^{\infty} q^{\frac{\ell(\ell-1)}{2}} z^{\ell}$$ (Jacobi triple product)

$$= \underbrace{e^{\pi i\tau/12} \cdot (\eta(\tau))^{-1}}_{\varphi} \cdot \theta\left(\zeta - \frac{\tau}{2} ; \tau\right) \text{ where } \begin{cases} q = \exp 2\pi i\tau \\ z = \exp 2\pi i\zeta . \end{cases}$$



Modular properties of η: (weight 1/2, level 1)

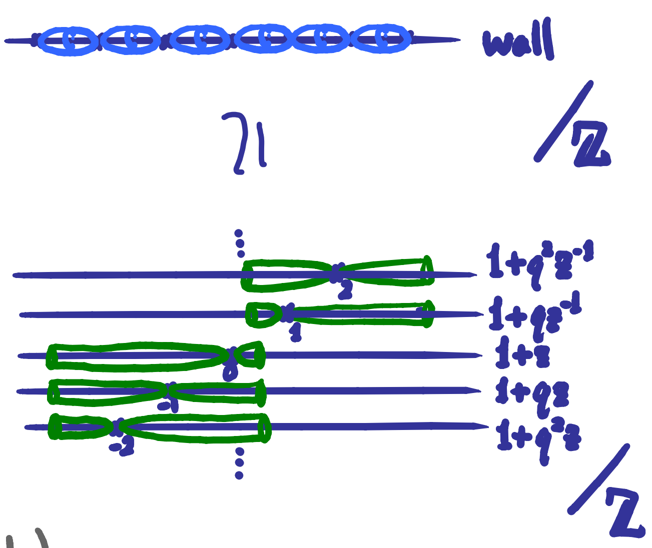
$$\begin{cases} \eta(\tau+1) = e^{\pi i/12}\eta(\tau) ; \\ \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau). \end{cases} \qquad \tau \in \mathcal{H}/SL(2,\mathbb{Z})$$

Modular properties of θ:

$$\theta(\zeta, \tau+1) = \theta(\zeta+\tfrac{1}{2}, \tau)$$

$$\theta(\zeta/\tau, \tfrac{-1}{\tau}) = (-i\tau)^{1/2} \exp\left(\frac{\pi}{\tau} i \zeta^2\right) \theta(\zeta, \tau)$$

SYZ mirror of $\tilde{A}_0$ surface



$$F^{\text{open}} = \prod_{j=1}^{\infty}(1+q^j z^{-1}) \cdot \prod_{k=0}^{\infty}(1+q^k z)$$

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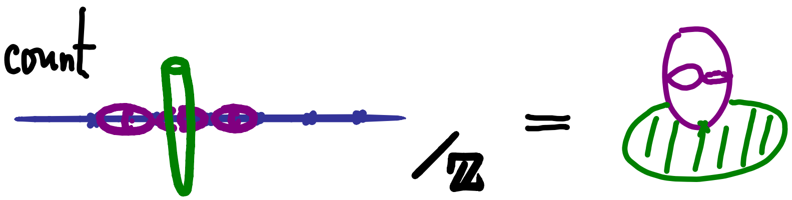

Modular properties of η : (weight ½, level 1)

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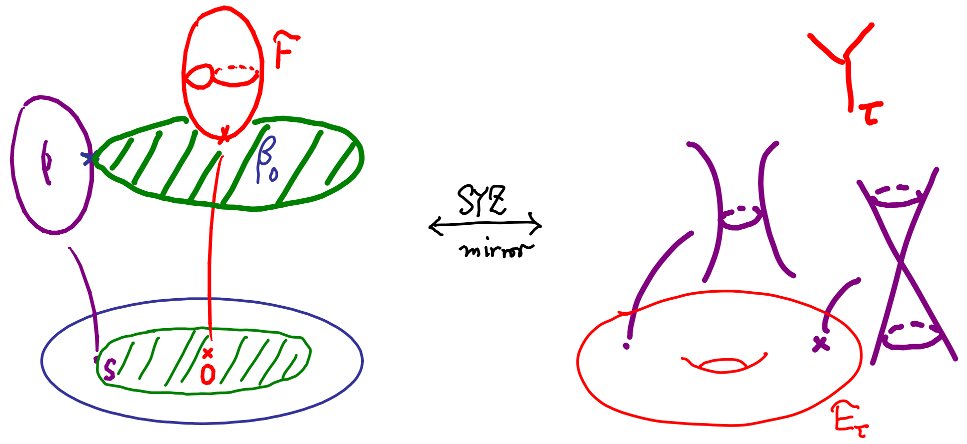
Cor. : The mirror family $Y_\tau = \{uv = F^{\text{open}}(z;q)\}$ extends to the global moduli $\mathcal{H}/SL(2,\mathbb{Z}) \ni \tau$.

SYZ mirror and root of Yau-Zaslow formula

The SYZ mirror of $\tilde{A}_0$ surface is the conic fibration

$$Y_\tau \triangleq \left\{(u,v,e^{2\pi i\zeta}) \in \mathbb{C}^2\times\mathbb{C}^\times : uv = \varphi(\tau)\cdot\underbrace{\Theta\left(\zeta-\tfrac{\tau}{2},\tau\right)}_{\text{Jacobi theta function}}\right\}\Big/\mathbb{Z}$$

$$(\zeta,\tau)\xmapsto{\overset{\rotatebox{90}{$\in$}}{k}}(\zeta+k\tau,\tau)$$



counting discs in class β_0+kF

where

$$\varphi(\tau)\triangleq\sum_{k\geq 0} n^{L}_{\beta_0+kF}\, e^{2k\pi i\tau} = \frac{e^{\pi i\tau/12}}{\eta(\tau)}.$$

root of Yau-Zaslow formula

A-side B-side

Geometric transitions ⟷ modularity

Yau-Zaslow formula

Theorem: [Beauville, Chen, Fantechi-Gottsche-van Straten, Bryan-Leung, Lee-Leung, Wu, Klemm-Maulik-Pandharipande-Scheidegger]

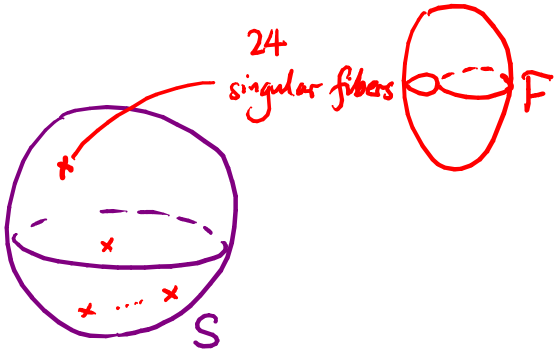
X: compact K3. $N(k.r) \triangleq$ # rational curves in class A where $A^2 = 2k-2$ & index $A = r$.

Then $\sum_{k>0} N(k.r)\, q^k = \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k}\right)^{24} = (\varphi(\tau))^{24}$.

Take $r=1$ (primitive) and $A = S+kF$:

$$\left(A^2 = \overbrace{S^2}^{-2} + 2k\, \overbrace{S\cdot F}^{1} + k^2 \overbrace{F^2}^{0} = 2k-2. \right)$$

$$\sum_{k>0} N_{S+kF}\, e^{k\pi i\tau} = (\varphi(\tau))^{24}. \qquad \left(\varphi(\tau) = \frac{e^{\pi i\tau/12}}{\eta(\tau)}.\right)$$



Yau-Zaslow formula

Theorem : [Beauville, Chen, Fantechi-Göttsche-van Straten, Bryan-Leung, Lee-Leung, Wu, Klemm-Maulik-Pandharipande-Scheidegger]

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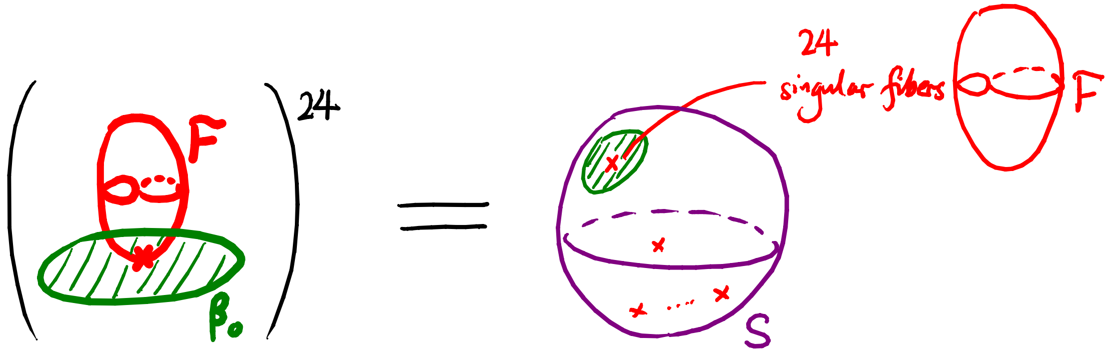
$$\sum_{k>0} N_{S+kF}\, e^{k\pi i\tau} = (\varphi(\tau))^{24}. \quad \left(\varphi(\tau) = \frac{e^{\pi i\tau/12}}{\eta(\tau)}.\right)$$

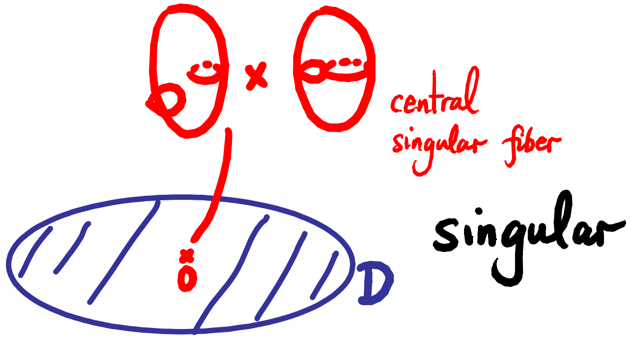
Our formula :

$$\sum_{k\geq 0} n^L_{\beta_0+kF}\, e^{2\pi i\tau} = \varphi(\tau).$$

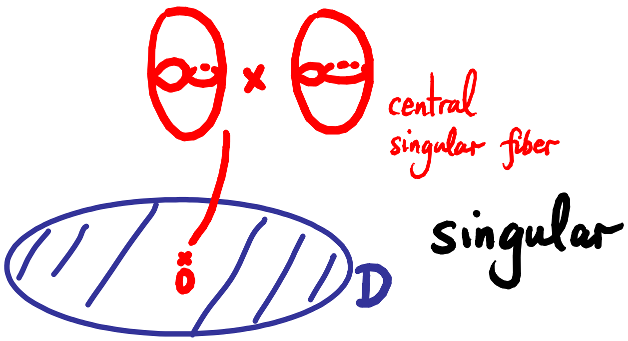


Fiber products of $\tilde{A}_0$ surfaces

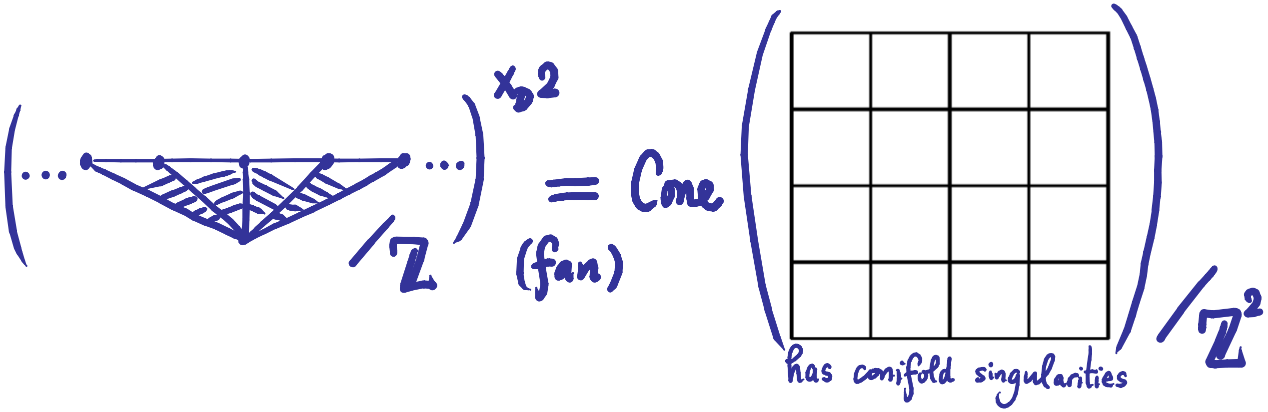
$$\tilde{A}_0 \times_D \tilde{A}_0$$

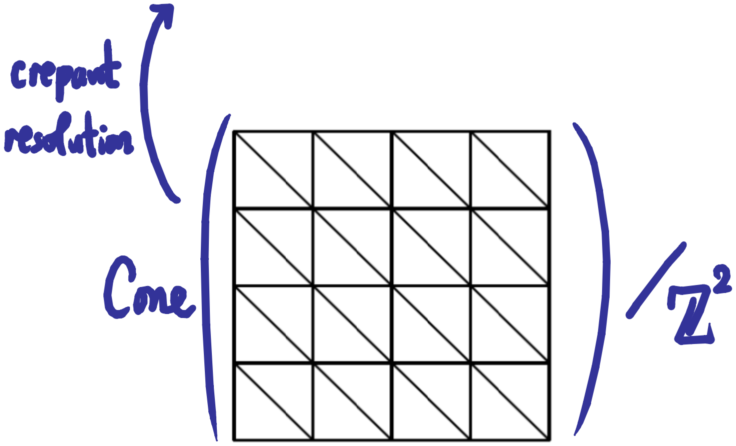


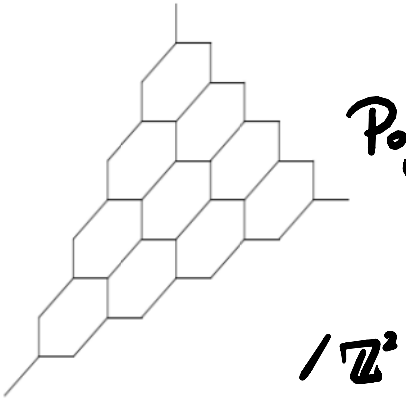
Fiber products of $\tilde{A}_0$ surfaces $\tilde{A}_0 \times_D \tilde{A}_0$



Toric realization

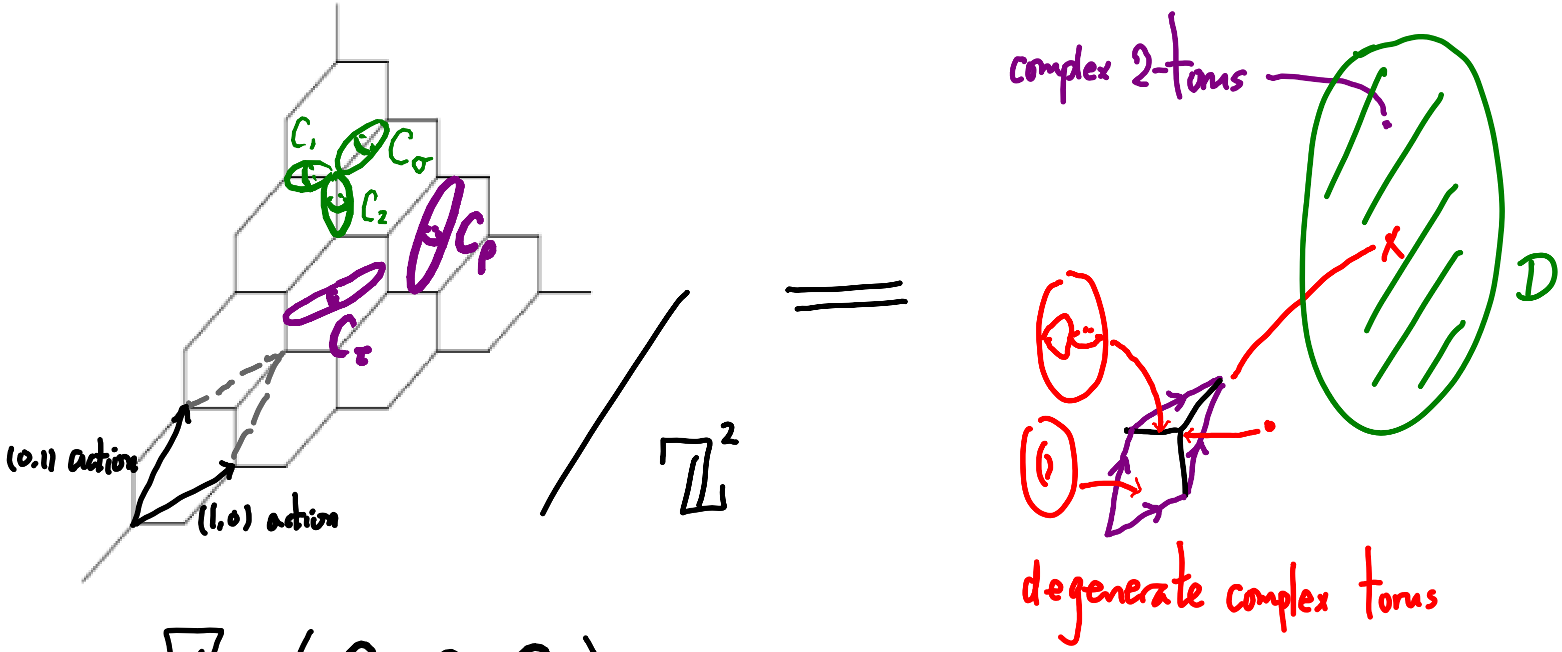






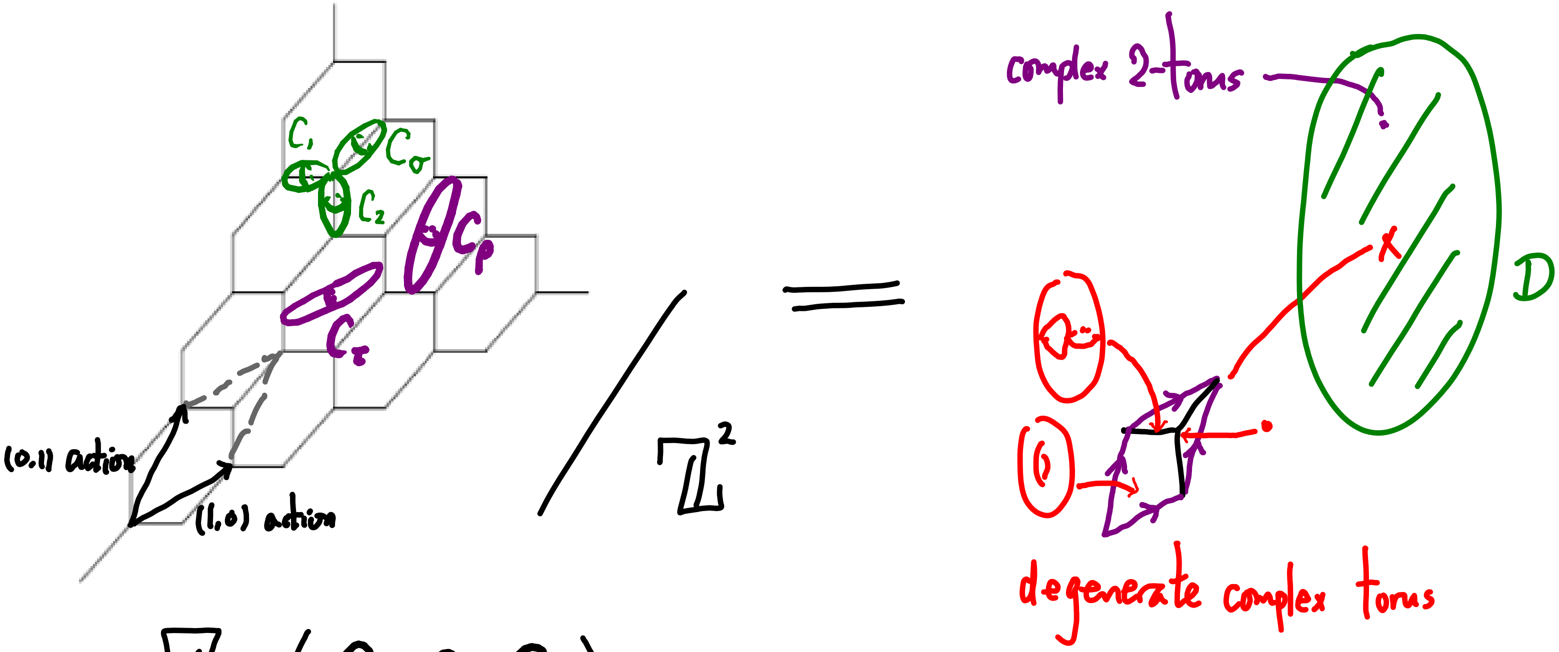
Polytope picture

SYZ mirror of $\tilde{A}_0 \times_D \tilde{A}_0$ threefold



Kähler cone $= \mathbb{Z}_{\geq 0}\langle C_1, C_2, C_\sigma \rangle$.

SYZ mirror of $\tilde{A}_0 \times_D \tilde{A}_0$ threefold



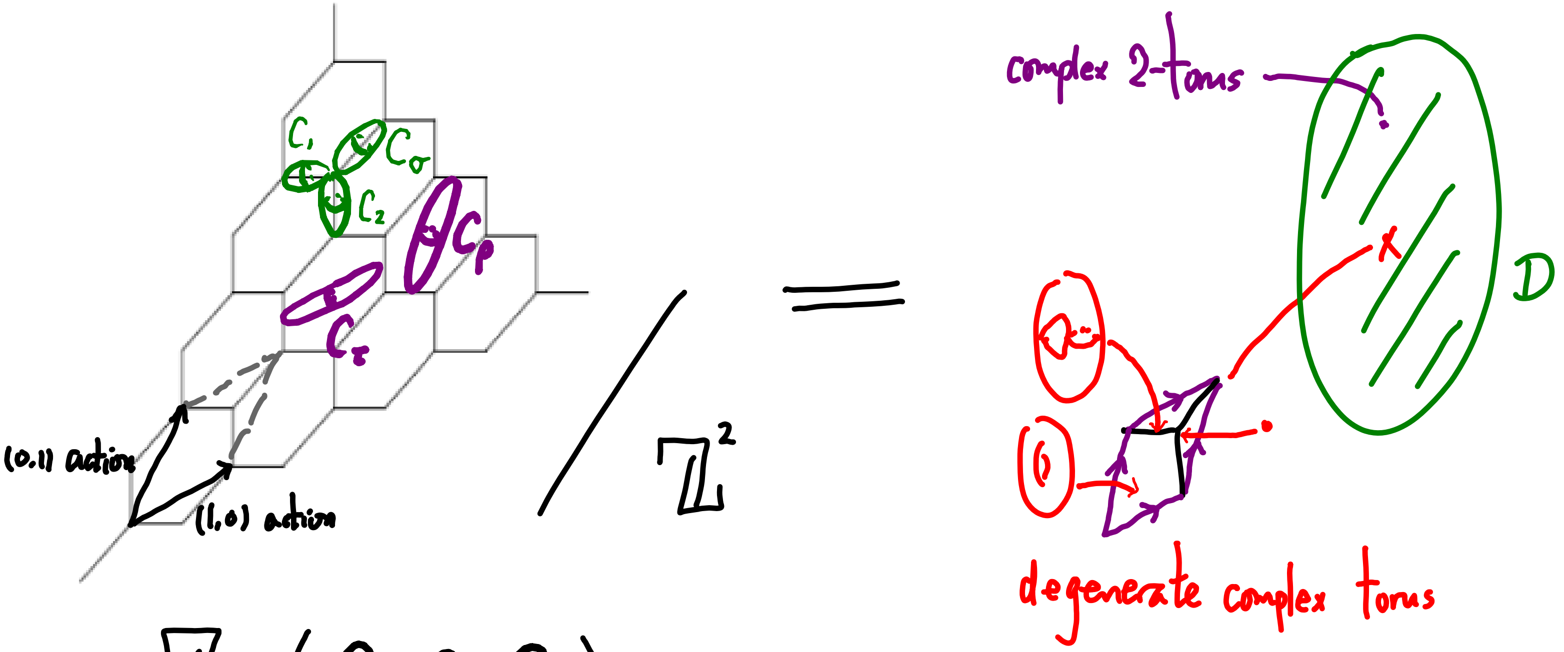
Kähler cone $= \mathbb{Z}_{\geq 0}\langle C_1, C_2, C_\sigma \rangle$.

Let $C_\tau = C_1 + C_\sigma$; $C_\rho = C_2 + C_\sigma$. $H_2(\mathbb{Z}) = \mathbb{Z}\langle C_\tau, C_\rho, C_\sigma \rangle$.

$\Rightarrow$ Three Kähler parameters: τ, ρ, σ. $q^\tau = \exp 2\pi i \tau$. ($\tau^{Im}$ is the area of C_τ.)

Note: $\tau^{Im} > \sigma^{Im} > 0$ and $\rho^{Im} > \sigma^{Im} > 0$.

SYZ mirror of $\tilde{A}_0 \times_D \tilde{A}_0$ threefold



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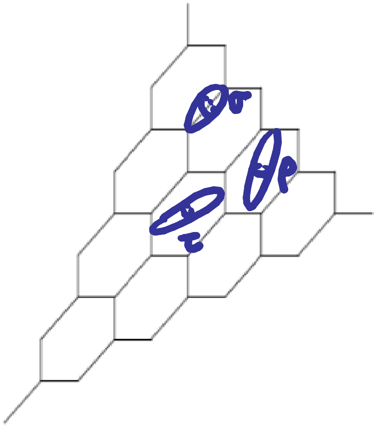
Can put this as $\Omega = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix}$ period matrix. Ω^{Im} is positive definite.

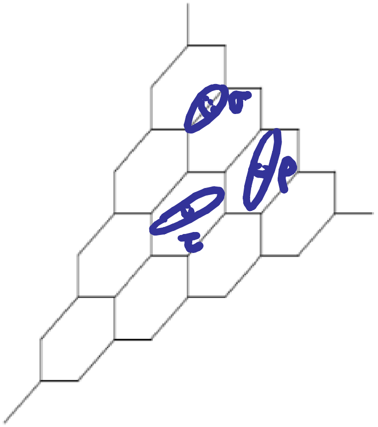
Siegal upper half space



Period matrix: $\Omega = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix}$.

$$\mathcal{H}_g \triangleq \{\Omega \in \mathrm{Sym}_{g \times g}(\mathbb{C}) : \mathrm{Im}\,\Omega \text{ positive definite}\}.$$

Siegal upper half space



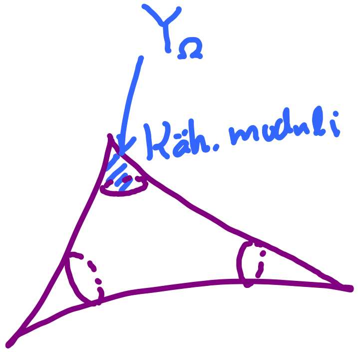
Period matrix: $\Omega = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix}$.

$Sp(2g.\mathbb{Z}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega \triangleq (A\Omega + B)\cdot(C\Omega + D)^{-1}$

$\downarrow$ Action generated by $\begin{pmatrix} A & \\ & (A^{-1})^t \end{pmatrix}$, $\begin{pmatrix} I & B \\ & I \end{pmatrix}$, $\begin{pmatrix} & -I \\ I & \end{pmatrix}$.

$\mathcal{H}_g \triangleq \{\Omega \in Sym_{g\times g}(\mathbb{C}) : \mathrm{Im}\,\Omega$ positive definite$\}$.

$\mathcal{H}_g / Sp(2g,\mathbb{Z})$ parametrizes Abelian varieties $\mathbb{C}^g / \langle (I.\Omega) \rangle$.



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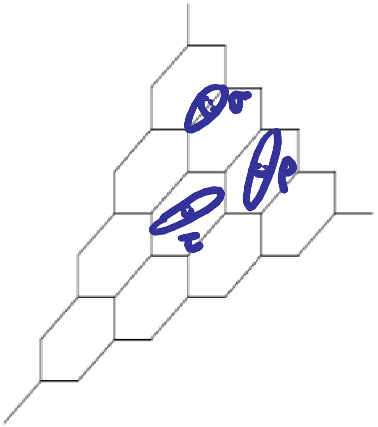

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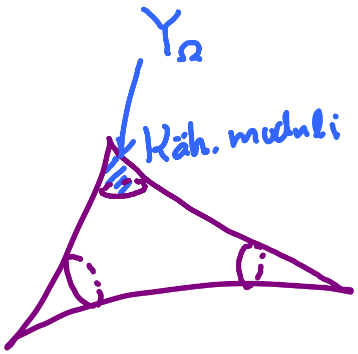
$\mathcal{H}_g / Sp(2g,\mathbb{Z})$ parametrizes Abelian varieties $\mathbb{C}^g / \langle (I.\Omega) \rangle$.

$$\begin{tikzcd} Y_\Omega = \{ uv = F^{open}(z;\Omega) \} \arrow[d] \\ \Omega \in \text{Kähler moduli} \underset{open}{\subseteq} \mathcal{H}_g / Sp(2g,\mathbb{Z}). \end{tikzcd}$$



Explicit expression

Theorem: [Kanazawa - L.]

The SYZ mirror of $\tilde{A}_0 \times_D \tilde{A}_0$ threefold is $\{uv = F^{open}\}/\mathbb{Z}^2$,

$$F^{open}(\zeta_1, \zeta_2 ; \Omega) = \underbrace{\left(\sum_{\alpha \in H_2^{eff}} n_{\beta_0+\alpha} q^{\alpha}\right)}_{\varphi(q)} \cdot \underbrace{\Theta_2 \begin{bmatrix} 0 & 0 \\ \frac{-\tau}{2} & \frac{-\rho}{2} \end{bmatrix} (\zeta_1, \zeta_2, \Omega)}_{\sum_{\vec{m} \in \mathbb{Z}^2} q_\tau^{\frac{m_1^2 - m_1}{2}} q_\rho^{\frac{m_2^2 - m_2}{2}} q_\sigma^{m_1 m_2} Z_1^{m_1} Z_2^{m_2}}.$$

Riemann theta function

Explicit expression

Theorem: [Kanazawa - L.]

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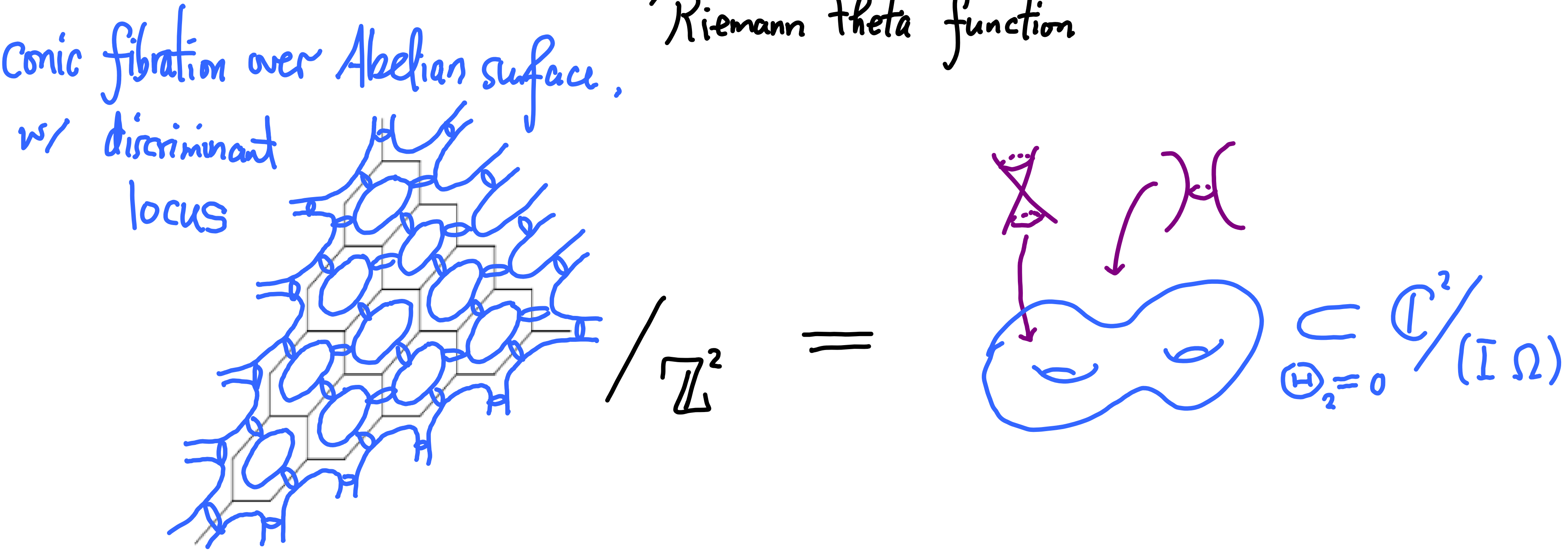
Riemann theta function



Modular properties of Riemann theta function

$Sp(2g.\mathbb{Z}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$\downarrow$ Action generated by $\begin{pmatrix} A & \\ & (A^{-1})^t \end{pmatrix}, \begin{pmatrix} I & B \\ & I \end{pmatrix}, \begin{pmatrix} & -I \\ I & \end{pmatrix}.$

$\mathcal{H}_g \triangleq \{\Omega \in Sym_{g\times g}(\mathbb{C}) : \mathrm{Im}\,\Omega \text{ positive definite}\}$, $\Theta : (\mathbb{C}^\times)^2 \times \mathcal{H}_g \longrightarrow \mathbb{C}$.

I. $\Theta\begin{bmatrix} 0 & 0 \\ -\frac{I}{2} & -\frac{\rho}{2} \end{bmatrix}(A\vec{\xi} ; A\Omega A^t) = \Theta\begin{bmatrix} 0 & 0 \\ -\frac{I}{2} & -\frac{\rho}{2} \end{bmatrix}(\vec{\xi} ; \Omega).$

II. $\Theta\begin{bmatrix} 0 & 0 \\ -\frac{I}{2} & -\frac{\rho}{2} \end{bmatrix}(\vec{\xi} ; \Omega + B) = \Theta\begin{bmatrix} 0 & 0 \\ -\frac{I}{2} & -\frac{\rho}{2} \end{bmatrix}(\vec{\xi} ; \Omega).$

III. $\Theta\begin{bmatrix} 0 & 0 \\ -\frac{I}{2} & -\frac{\rho}{2} \end{bmatrix}(\vec{\xi} ; -\Omega^{-1}) = \sqrt{\det(-i\Omega)} \cdot e^{\pi i (\xi - v)^t \Omega (\xi - v)} \cdot \Theta\begin{bmatrix} 0 & 0 \\ -\frac{I^2}{2}-\frac{\sigma\rho}{2} & -\frac{\rho I}{2}-\frac{\rho^2}{2} \end{bmatrix}(\Omega\vec{\xi}, \Omega).$

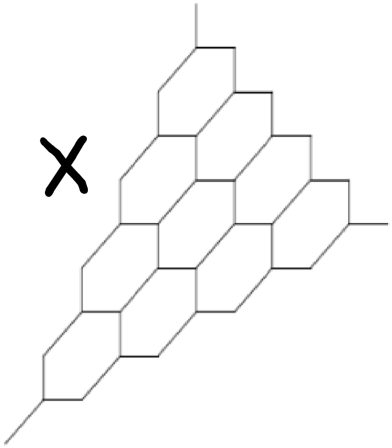
Cor. : The mirror family $Y_\Omega = \{uv = \varphi(\Omega) \cdot \Theta(\vec{\xi} ; \Omega)\}$ extends over the global moduli $\mathcal{H}_g / Sp(2g.\mathbb{Z})$.

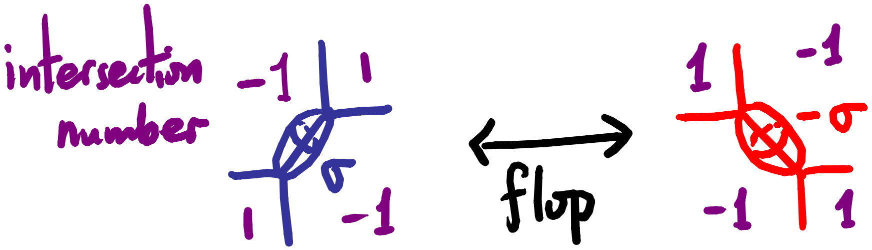
How do these properties come up from mirror geometry?

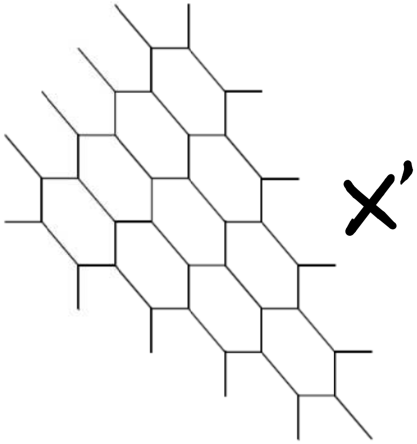
Modular property I

Consider $\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \cdot \Omega = A \cdot \Omega \cdot A^t$ for $A \in GL_2(\mathbb{Z})$.

e.g. $A = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} : \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix} \longmapsto \begin{pmatrix} \tau & -\sigma \\ -\sigma & \rho \end{pmatrix}$



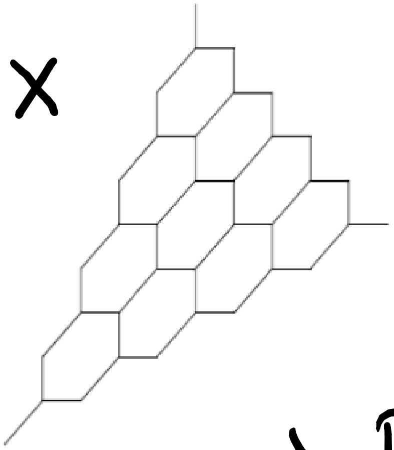




$$\hat{F}^{open}_{X}\left(\vec{\xi}\,;\begin{pmatrix} \tau & -\sigma \\ -\sigma & \rho \end{pmatrix}\right) = \hat{F}^{open}_{X'}\left(\vec{\xi}\,;\begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix}\right).$$

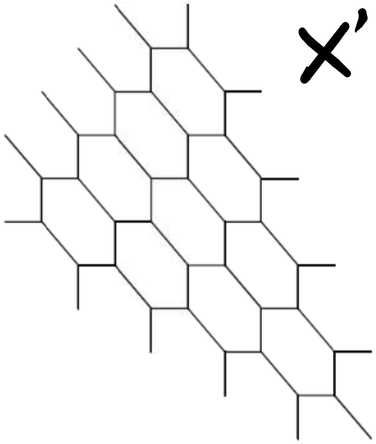
Modular property I

X



$$F^{open}_{X}\left(\vec{\xi}\,;\begin{pmatrix}\tau & -\sigma\\ -\sigma & \rho\end{pmatrix}\right)=F^{open}_{X'}\left(\vec{\xi}\,;\begin{pmatrix}\tau & \sigma\\ \sigma & \rho\end{pmatrix}\right).$$

X'



Also $X \xleftrightarrow{\begin{pmatrix}-1 & \\ & 1\end{pmatrix}} X'$

$$\Rightarrow F^{open}_{X}\left(-\zeta_1,\zeta_2;\begin{pmatrix}\tau & \sigma\\ \sigma & \rho\end{pmatrix}\right)=F^{open}_{X'}\left(\zeta_1,\zeta_2;\begin{pmatrix}\tau & \sigma\\ \sigma & \rho\end{pmatrix}\right).$$

$$\therefore\ F^{open}_{X}\left(\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\vec{\xi}\,;\Omega\right)=F^{open}_{X}\left(\vec{\xi},\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\Omega\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\right).$$

i.e. $F^{open}_{X}\left(\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\vec{\xi}\,;\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\Omega\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\right)=F^{open}_{X}(\vec{\xi},\Omega).$

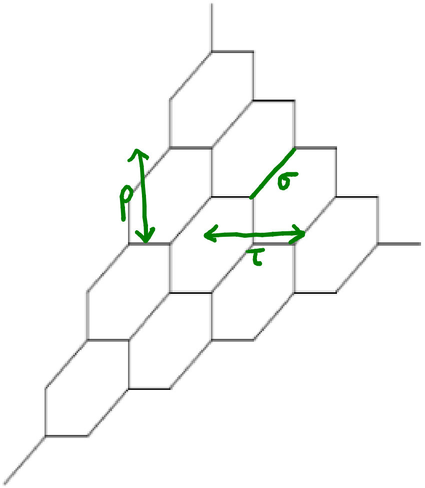
Similar consideration works for $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.

$$\Rightarrow\ F^{open}(A\vec{\xi},A\Omega A^{t})=F^{open}(\vec{\xi},\Omega).$$

Modular property II

Consider $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \cdot \Omega = \Omega + B$ for $B \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$.

$$\Omega = \begin{pmatrix} \tau & \sigma \\ \sigma & \rho \end{pmatrix}.$$



$q_\tau = \exp 2\pi i \tau = \exp 2\pi i (\tau + b)$ for $b \in \mathbb{Z}$.

and similar for q_σ, q_ρ.

$$F^{open} = \sum_{\vec{m} \in \mathbb{Z}^2} \Big(\sum_{\alpha \in H_2^{eff}} n_{\beta_{\vec{m}}+\alpha} q^{\alpha} \Big) q^{C_{\vec{m}}} \vec{z}^{\vec{m}},$$

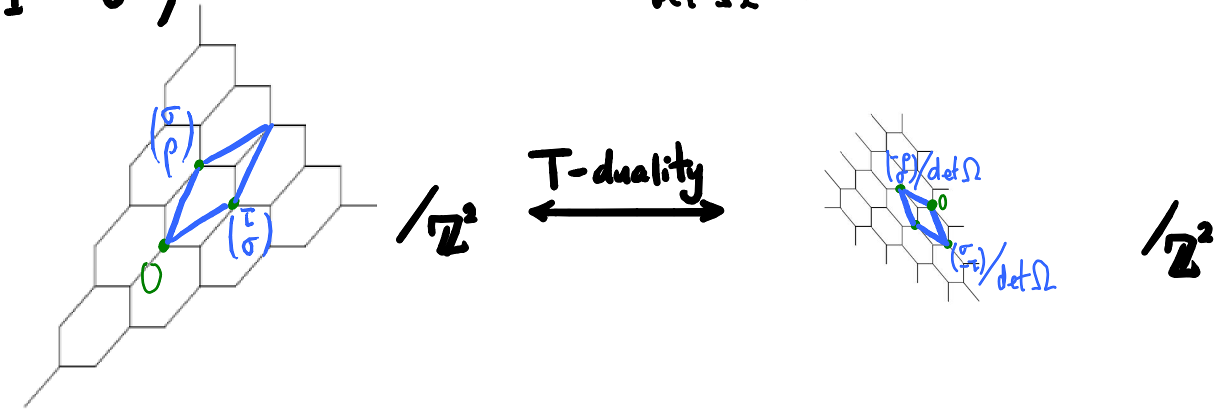
$q^{C_{\vec{m}}}, q^{\alpha}$ takes the form $q_\tau^{\alpha_\tau} q_\rho^{\alpha_\rho} q_\sigma^{\alpha_\sigma}$ for $\alpha_\tau, \alpha_\rho, \alpha_\sigma \in \mathbb{Z}$.

Hence

$$F_X^{open}(\vec{\xi}; \Omega + B) = F_X^{open}(\vec{\xi}; \Omega).$$

Modular property III: T-duality on base

Consider $\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\cdot\Omega = -\Omega^{-1} = \frac{1}{\det\Omega}\begin{pmatrix}-\rho & \sigma\\ \sigma & -\tau\end{pmatrix}$, $\Omega = \begin{pmatrix}\tau & \sigma\\ \sigma & \rho\end{pmatrix}$.



Do not preserve $\Omega^{Im} \sim \infty$!

$$\widetilde{F}^{open}(\zeta_1,\zeta_2;\Omega) = \underbrace{\Big(\sum_{\alpha\in H_2^{eff}} n_{\beta_0+\alpha}\, q^{\alpha}\Big)}_{\varphi}\cdot \Theta_2\begin{bmatrix}0 & \\ \frac{-\tau}{2} & \frac{-\rho}{2}\end{bmatrix}(\zeta_1,\zeta_2,\Omega).$$

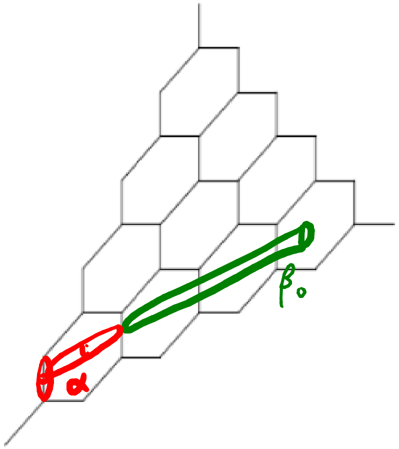
high dimension analog of $\frac{e^{\pi i\tau/12}}{(\tau)}$

$\widetilde{F}^{open}(\zeta_1,\zeta_2;-\Omega^{-1}) \sim \widetilde{F}^{open}(\zeta_1,\zeta_2;\Omega)$. Still mysterious!

Want to understand better by Bridgeland stability.

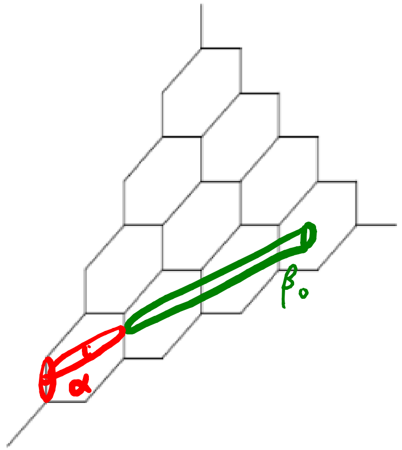
Gross-Siebert normalization condition

To compute $\varphi(q) = \left(\sum_{\alpha \in H_0^{\mathrm{eff}}} n_{\beta_0+\alpha} q^{\alpha}\right)$:



Do not have product formula in general dimensions.

Gross-Siebert normalization condition



To compute $\varphi(q) = \left(\sum_{\alpha \in H_0^{\text{eff}}} n_{\beta_0+\alpha}\, q^{\alpha}\right)$:

Do not have product formula in general dimensions.

Theorem: [L.]

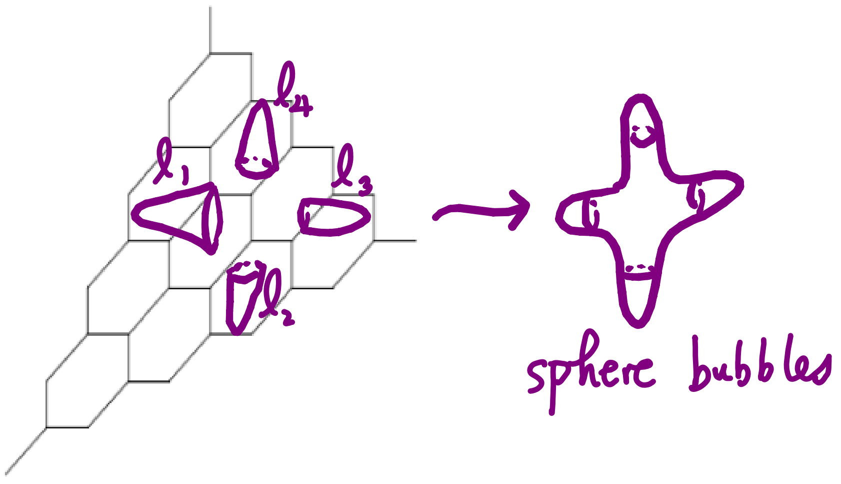
F^{open} satisfies the Gross-Siebert normalization:

$$\vec{z}^{\,0}\text{-coefficient of } \log \underbrace{\tilde{F}^{\text{open}}(\vec{z};q)}_{\underbrace{\left(\sum_{\alpha \in H_0^{\text{eff}}} n_{\beta_0+\alpha}\, q^{\alpha}\right)}_{\varphi(q) = 1 + o(q)} \cdot \underbrace{\sum_{\vec{m} \in \mathbb{Z}^2} q_\tau^{\frac{m_1^2-m_1}{2}}\, q_\rho^{\frac{m_2^2-m_2}{2}}\, q_\sigma^{m_1 m_2}\, z_1^{m_1} z_2^{m_2}}_{\Theta_2\begin{bmatrix} 0 & 0 \\ \frac{-\tau}{2} & \frac{-\rho}{2} \end{bmatrix}(\zeta_1,\zeta_2)}} \text{ is independent of } q.$$

The open GW generating function

$$\log\varphi = \vec{z}^{\,0} \text{ coefficient of } \underbrace{-\log \sum_{\vec{m}\in\mathbb{Z}^2} q_\tau^{\frac{m_1^2-m_1}{2}} q_\rho^{\frac{m_2^2-m_2}{2}} q_\sigma^{m_1m_2} z_1^{m_1} z_2^{m_2}}_{-\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\left(\sum_{\vec{m}\in\mathbb{Z}^2-\{0\}} q_\tau^{\frac{m_1^2-m_1}{2}} q_\rho^{\frac{m_2^2-m_2}{2}} q_\sigma^{m_1m_2} z_1^{m_1} z_2^{m_2}\right)^p}$$

$$\varphi = \exp\left(\sum_{p\geq 2}\frac{(-1)^p}{p}\sum_{\substack{(\mathbf{l}_i=(l_i^1,l_i^2)\in\mathbb{Z}^2\setminus 0)_{i=1}^p\\ \textit{with } \sum_{i=1}^p \mathbf{l}_i=0}} \exp\left(\sum_{k=1}^p \pi i \mathbf{l}_k\cdot\Omega\cdot\mathbf{l}_k^T\right)\right).$$

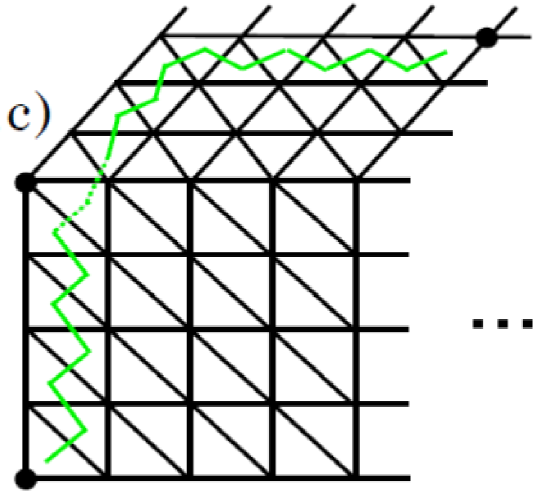


Higher dim. analog of $\frac{q^{1/24}}{\eta(q)}$.

Expect it is a Siegel modular form.

Higher dimensions: $\tilde{A}_{d_1-1} \times_D \cdots \times_D \tilde{A}_{d_\ell-1}$.

fan



Theorem [Kanazawa - L.] :

$\tilde{A}_{d_1-1} \times_D \cdots \times_D \tilde{A}_{d_\ell-1}$ $(\ell+1)$-fold is SYZ mirror to

$\{uv = F^{\text{open}}(z;q)\}$, where

$$F^{\text{open}} = \sum_{a_1,\dots,a_l=0}^{d_1-1,\dots,d_l-1} K_{(a_1,\dots,a_l)} \cdot \Delta_{(a_1,\dots,a_l)} \cdot \Theta_{l,}^{(a_1\dots a_\ell)}$$ on Abelian variety $\mathcal{A} \triangleq \mathbb{C}^n/\langle I\ \Omega\rangle$.

$$\Theta_\ell^{(a_1\dots a_\ell)} = \Theta_l \begin{bmatrix} \left(\frac{a_1}{d_1},\dots,\frac{a_l}{d_l}\right) \\ \left(\frac{-d_1\tau_1}{2} + \sum_{k=0}^{d_1-1} k\tau_{1,(-1-k,0,\dots,0)},\dots,\frac{-d_l\tau_l}{2} + \sum_{k=0}^{d_l-1} k\tau_{l,(0,\dots,0,-1-k)}\right) \end{bmatrix} (d_1\cdot\zeta_1,\dots,d_l\cdot\zeta_l;\Omega)$$

basis of $(d_1 \dots d_\ell)$ polarization on $\mathcal{A}$.
(ample line bundle)

Mirrors of general-type varieties

$$\begin{tikzcd} & \text{conic fibration} \arrow[d] & \\ \tilde{A}_{d_1-1} \times_{D} \cdots \times_{D} \tilde{A}_{d_\ell-1} \arrow[r, leftrightarrow, "\text{mirror}"] & A & \supset Y \end{tikzcd}$$

degeneracy locus

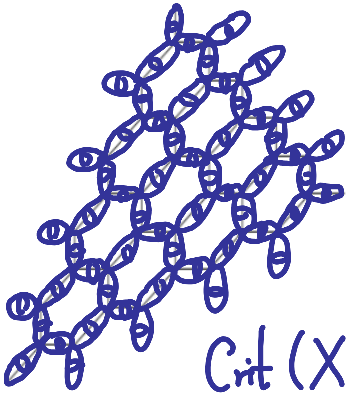
divisor of $(d_1, \ldots, d_\ell)$ polarization

Mirrors of general-type varieties

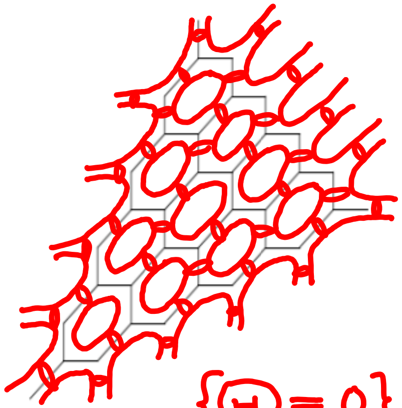
$$\begin{tikzcd} \tilde{A}_{d_1-1}\times_D\cdots\times_D\tilde{A}_{d_\ell-1} \arrow[r, leftrightarrow, "\text{mirror}"] & \text{conic fibration} \arrow[d] \\ & \mathcal{A} \underset{\text{degeneracy locus}}{\supset} Y \end{tikzcd}$$

Y divisor of $(d_1,\dots,d_\ell)$ polarization

$$\mathrm{Crit}.\left(\begin{array}{c} \tilde{A}_{d_1-1}\times_D\cdots\times_D\tilde{A}_{d_\ell-1} \\ \downarrow \\ D \end{array}\right) \xleftrightarrow{\text{mirror}} Y$$



$\mathrm{Crit}(X_\Sigma, w) / \mathbb{Z}^2$



$\{\Theta = 0\} \subset (\mathbb{C}^*)^2 / \mathbb{Z}^2$

Mirrors of general-type varieties

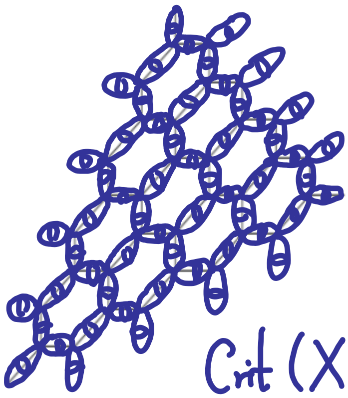
$$\begin{tikzcd} & \text{conic fibration} \arrow[d] \\ \tilde{A}_{d_1-1} \times_D \cdots \times_D \tilde{A}_{d_\ell-1} \arrow[r, leftrightarrow, "\text{mirror}"] & \mathcal{A} \supset Y \end{tikzcd}$$

degeneracy locus

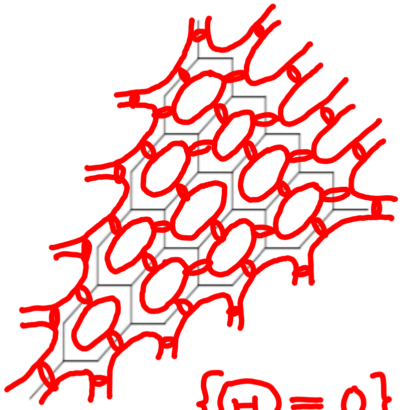
Y divisor of $(d_1, \dots, d_\ell)$ polarization

$$\mathrm{Crit.}\left(\begin{tikzcd} \tilde{A}_{d_1-1} \times_D \cdots \times_D \tilde{A}_{d_\ell-1} \arrow[d] \\ D \end{tikzcd}\right) \xleftrightarrow{\text{mirror}} Y$$

Remark: Reconstruct (X_Σ, W) from Lagrangian immersions in Y.

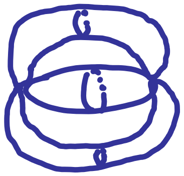


$\mathrm{Crit}(X_\Sigma, w) \, / \mathbb{Z}^2$



$\{\Theta = 0\} \subset (\mathbb{C}^\times)^2 \, / \mathbb{Z}^2$

e.g.



$\xleftrightarrow[\text{[Seidel]}]{\text{mirror}}$